Transition to Slower Population Growth: Demography and its Effect on Real Interest Rates

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Abstract

The past 30 years has witnessed a worldwide decrease in real interest rates. We demonstrate that a large part of the fall in interest rates can be explained by changes in demography, which are as the result of a sudden fall in fertility rates across all of the advanced economies in the early 1970s. In the long run this leads to a lower population growth rate. In the short run the cohort born just before the fertility shock is disproportionally larger than the cohorts born before and after. As this large cohort accumulates assets for retirement, their savings flood the capital market leading to a collapse in its return and hence a collapse in interest rates. Our model predicts that real interest rates will continue to fall, overshooting the new balanced growth path level, until hitting a trough around the year 2035.

1 Introduction

The fall in real interest rates experienced in the past several decades has been an area of intense academic debate in recent years. Real interest rates have been negative in the United States and the Euro-zone since 2008. This has important implications for returns on investment, financial stability, and also the operation of monetary policy subject to the zero lower bound, see Teulings and Baldwin (2015) for an overview of the secular stagnation debate. In this paper we show that changes in demographic structure, due a collapse in fertility rates across all of the advanced economies in the early 1970s, can explain a large part of the fall in interest rates we witnessed. The cohort born just before the fertility shock is far greater in size than the cohorts born before and after. As a result, the age composition of the population today deviates strongly from the composition in a world of constant population growth. This leads to a disruption in the life-cycle saving patterns of the overlapping generations, and the savings of the large cohort drives down the return of capital and hence interest rates.

Across all developed economies, there was a sharp decline in fertility rates by about 30% or so around the early 1970s (see our discussion in section 2). In the U.S. the decline in fertility rates can be explained by the contraceptive pill becoming widely available towards the late 1960s, as well as increasing career and higher education opportunities for women, see Goldin and Katz (2002). The fall in fertility had corresponding effects on cohort size, with the cohorts born afterwards being much smaller in size. Some 20 to 25 years later, the event had an echo effect when the first cohort of mothers born after the fertility shock were supposed to give birth to the next generation. In Germany since these post-shock cohorts were 30% smaller than their pre-shock predecessors, cohort size dropped again by some 30%. As a consequence, German cohorts born after 1990 are less than half the size of the large cohort born before 1970, although exact numbers differ somewhat between countries depending on the exact pattern in fertility, the age of giving birth and migration. Since the population was growing rather rapidly in most countries in the decades before, the large cohort born between 1964 and 1968 are far greater in size than the cohorts born both before and after. As we transition to a future with slower population growth, the current age composition of population deviates strongly from the composition in any Balanced Growth Path (BGP) equilibrium, whether a pyramid (in the case of a growing population) or an inverted pyramid (in the case of a shrinking population). This transitional composition can account for a sharp decline in real interest rates, one even that overshoots the new BGP level. Note that due to the close synchronicity of demography across the advanced economies, international capital markets may offer little help to the problem of low interest rates. Furthermore while developing economies suffered a lesser fall in fertility rates, there are greater frictions and greater risks in the movement of capital to those countries. Nonetheless it is sensible to think broadly of a global capital market with a global real interest rate. While our analysis is best calibrated to a single country, its implications are easily extended to a wider global scale.

Because cohorts seek to smooth consumption across their life cycles, a cohort saves during the years it is active on the labor market to accumulate assets in order to finance their consumption in retirement. The desired asset holdings by a generation is at its peak around the age of retirement (around 65 in most countries). In a BGP, this life-cycle pattern in the asset accumulation of a cohort is smoothly accommodated by the other cohorts which are in a different stages of their life cycles. The large cohort born before the negative fertility shock of the early 1970s, currently aged between 45 and 49, disrupts this accommodating process. This large cohort is accumulating assets to finance their future consumption, while the usual absorbers of these savings, the retirees and the youngsters, are in short supply. Our analysis shows that the saving of this large cohort can explain the pattern of falling real interest rates. Our model predicts negative real interest rates in 2016 as is observed empirically. Further, our model suggests that this downward trend will continue for another decade and half, overshooting the new BGP level until hitting a trough around the year 2035. After 2035 the large cohort begins to retire and deplete their savings, thus reversing the trend of high savings. Eventually real interest rates will recover somewhat before tending towards the new (lower) BGP level. Note that for Japan, there is an earlier large cohort - those born in the baby boom immediately after the Second World War. While a baby boom is common to most advanced economies following the Second World War, Japan's baby boom was extraordinary in terms of its intensity. Japan's baby-boom cohort, at around the age of 70 today, is so numerous that they dominate Japan's age distribution even in the year 2016. For this reason, the key features of the Japanese age distribution leads that of the western world by about 15 years. Our analysis when applied to Japan will likewise occur 15 years earlier and indeed this matches their experience. In this sense, Japan's experience over the past two decades provides a good laboratory for other advanced economies whose large cohorts are 15 years younger.

Samuelson (1958) was the first to derive a BGP relation between the growth rate g and the return to capital r in an economy with overlapping generations of workers and retirees. It turns out that an equilibrium real interest rate of g maximizes the welfare of the representative household, however this equilibrium may not always be attainable. Nonetheless, generally there is a close positive relationship between r and g in equilibrium. The fertility shock of the early 1970s reduced the long-run growth of population, g, which shall eventually lead to a lower return to capital r. What is more interesting however is the transition to the new BGP. The cohort size distribution carries memory and generates persistence; the enormous size of the large cohort born before the fertility shock makes our demography today particularly biased towards saving. As this large cohort builds up its stock of assets to finance their future consumption, the supply of savings greatly exceeds its demand, resulting in a fall in the real interest rate that overshoots the new BGP level. Although this overshooting transitory, it is of first-order importance as it lasts for several decades. The lower the elasticities of intertemporal substitution in consumption and of substitution between capital and labor, the larger this effect. The empirical literature suggests these elasticities to be less than unity, in the order of magnitude of a half (for the capital-labor elasticity of substitution see Chirinko (2004), Chirinko (2008), and Havránek (2015)). Using these values, our calculations show that after an initial increase at the arrival of the shock, r falls continually to a minimum that is about 2% below the new BGP level. The trough in real interest rates occurs around 2035, some 65 years after the fertility shock just as the large cohort begins to deplete their assets.

Several past studies have looked to investigate the effect of demographics on real interest rates. Rachel and Smith (2015) find that changes in demography has resulted in changes to the support ratio, the ratio of workers to the overall population. They find that an increase in the support ratio can explain an increase in savings supply accounting for a 0.9% fall of real interest rates. Carvalho, Ferrero and Nechio (2016) make a similar argument and find that they can account for a reduction in the natural rate of interest of 1.5%. Goodhart and Erfurth (2014) consider future support ratios and project that support ratios will fall in the future, thereby reversing the period of high savings and low interest rates. They predict that by 2025 real interest rates shall return to their historical norm of 2.5-3%. Teulings and Gottfries (2015) argue that the increase in life expectancy not offset by an increase in the retirement age has raised savings supply and hence reduced interest rates. While the above studies vary in their approach, their analyses all have in common the limitation of being static in some fashion. Consider an analysis based on the level of the support ratio, which argues that within a steady state equilibrium high support ratios correspond to low interest rates. Although this steady state argument broadly captures the direction of the demographic change, it misses the magnitude and the dynamics of the demographic transition. Specifically our analysis finds that the most important demographic factor for interest rates in the medium-run is the age *distribution*. While support ratios broadly captures the aging of the population, it does not account for the *distribution* of age within the labor force. It is precisely the concentration of mass in towards the older part of the labor force that generates our results. The implication is a much larger response for real interest rates, which would otherwise be missed.

As we focus our attention to the effect of demography, it is relatively innocuous to assume no technological progress in our modelling. Hence our g is simply the growth rate of population. To include a constant rate for technological progress would be equivalent to having faster rate of effective labor growth. Our results would be broadly unchanged with this inclusion, and the difference would be an increase in g and hence also an increase in r by a similar margin. Gordon (2012) in his study of historical U.S. growth makes the argument that innovations of recent decades are comparatively minor when compared to the innovations of the industrial revolution. He attributes this to the slowing productivity growth in recent decades. His findings imply a downwards pressure on per capita output growth through the unobserved technology term, as well as a corresponding negative effect on interest rates. We view this as complementary to our findings on the role of demography. Together the two negative forces can compound, leading to an altogether larger collapse in the real interest rate of the order we observe. Gordon's perspective fits into the broader discussion of long-term stagnant growth. Particularly relevant to low interest rates, Summers (2013) makes the argument that a persistently low natural rate of interest, the real interest rate consistent with full output, can lead to a recession of indefinite duration due to the Zero Lower Bound (ZLB) on policy rates. He terms a recession of this kind secular stagnation, see Eggertsson and Mehrotra (2014) for a model of secular stagnation. The basic argument goes, when the natural rate of interest is sufficiently negative, monetary policy cannot sufficiently lower the nominal interest rate due to the ZLB, leading to a mismatch between the real interest rate delivered versus the real interest rate desired (i.e. the natural rate of interest). This mismatch causes a recession, which persists as long as the natural rate of interest remains negative. In our model without nominal rigidities, the equilibrium real interest rate maps to the natural rate of interest in an economy with rigidities. Our analysis finds much support for Summers' hypothesis, and as is observed in the simulation for Germany, we see that a negative natural rate of interest may not be so unexpected due to demography alone.

The structure of the remainder of this paper is as follows. Section 2 presents a brief discussion on the general demographic trends common to all of the advanced economies, with a particular emphasis on the four largest economies namely the United States, China, Japan, and Germany. We propose a model of their demographic transitions and demonstrate that it accurately matches the age distribution of Germany in 2016. In section 3, we present a stylized model to explain the effect of the large cohort's savings on real interest rates. We find that real interest rates fall as the savings of the large cohort floods the capital market. In section 4 we present a large overlapping generations model, which takes our demographic model from section 2 and sets it in a general equilibrium framework with optimizing agents and a competitive capital market. We calibrate this model to Germany and present the resulting real interest rate transition path. We conclude in section 5.

2 Demographic transition since the 1970s

Figure 1 provides the age pyramids for the world's four largest economies, the United States, China, Japan, and Germany, in 2016.[1]

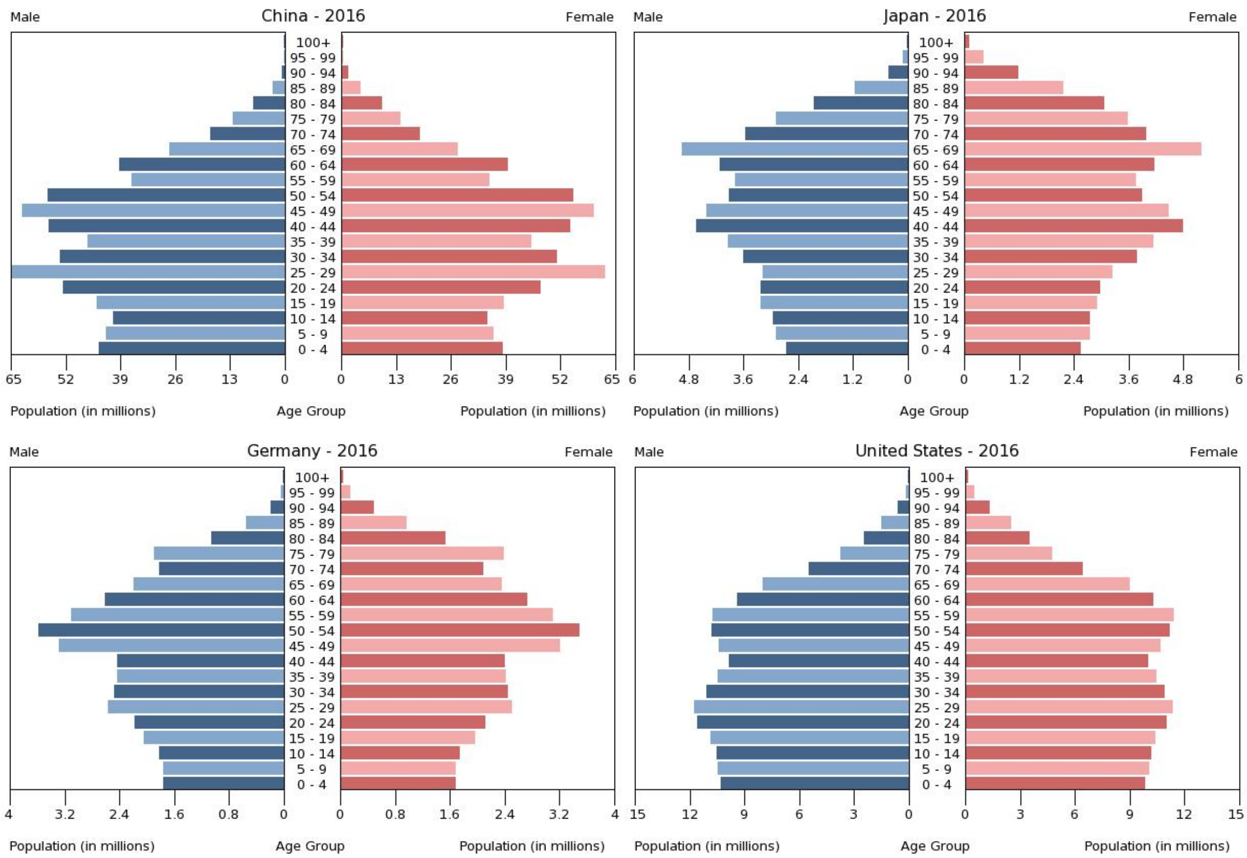

Figure 1: Age pyramids of the world's four largest economies.

All four graphs reveal a clear difference compared to the traditional graphs that applied 50 years ago and that do still apply for many developing countries: the age pyramid been replaced by a diamond (with more mass in the middle section). However at a more detailed level, the graphs also reveal clear differences between countries. For the sake of our discussion, we ignore the tops of the pillars above the age of 65, which largely reflects the effect of mortality at old age. Instead, we focus on the size of cohorts at the lower ages, which largely reflects differences in the number of births over time, and also subsequent migration

[1]Figure 1 is composed from figures taken from https://www.cia.gov/library/publications/the-world-factbook/fields/2010.html

flows. First, we may identify the relative size of the large cohort born before the negative fertility shock of the early 1970s, who are aged around 45-49 in 2016. In each country we see that due to a fall in fertility rates, there is a corresponding fall in cohort size as we move ahead in time, although the depth of this fall varies from country to country. The graph for the United States shows that they experience the smallest drop in cohort size following the fertility shock. We see that the biggest cohort (age 20-24) is only about 10-15 % larger than the smallest (age 30-39). The fall in fertility is offset by an inflow of new migrants, and hence the shape of the American age pyramid is therefore largely consistent with a stable population a growth rate of zero.

This is not the case in the other three countries, where roughly speaking, the bottom of the traditional age pyramid has been replaced by an inverted pyramid, resulting in the diamond shape we described earlier. However, there are clear difference even between these three countries. Japan stands out as its oldest large cohort (age 60-69) lead the largest cohorts in China (age 40-49) and Germany (age 45-54) by 20 and 15 years respectively. For Japan, their large cohort is those who were born in the baby boom immediately after the Second World War. In all three countries, there is an echo effect of the elderly large cohort some 20 to 25 years later, when the large cohort of fertile women gave rise to a boom in newborn babies. In Japan today, this echo cohort (in particular age 40-44) is larger than the cohort of their parents due to mortality at the higher ages.

Germany stands out as the country among the four with the most clear diamond shape. Following the fertility shock, the Total Fertility Rate (TFR) defined as the expected number of children a woman would have over her lifetime, dropped by some 30%. Again there is an echo effect some 20 to 25 years later, when the cohort of mothers declines by 30% due to the drop in fertility two decades before, leading to a further 30% decline in cohort size. From then on, cohort size keeps shrinking steadily, since the fertility rate is below the reproduction rate. Looking in 2016, Germany's smallest cohort is less than 50% the size of the largest cohort. We claim that the demographic transition experienced in each of these countries, particularly so for Germany, can be explained by a simple model of demographics where a country suffers a fertility shock around the year 1970.

2.1 A model of demographic transition

Suppose that households live for a total of J years (periods). Define the size of the cohort born at period t to be N_t. The total population of all those alive at period t is P_t, which is given by

$$P_t = \sum_{i=0}^{J-1} N_{t-i}. \tag{1}$$

Those between the ages of $\underline{F}$ and $\overline{F}$ are fertile. They determine the size of the newborn cohort according to

$$N_t = b_t \sum_{i=\underline{F}}^{\overline{F}-1} N_{t-1-i}, \tag{2}$$

where b_t is the birth rate at time t.

From these equations, we may derive the constant rate of population (and cohort size) growth for a given fixed birth rate b. By equation 1, we see that if population grows at a constant rate of g_P and cohort size grows at a constant rate of g_N, then it must be the case that $g_P = g_N$. From here we shall refer to this common growth rate as simply g. Rewriting equation 2 to

$$b = \frac{1}{\sum_{i=\underline{F}}^{\overline{F}-1} (1+g)^{-i}}, \tag{3}$$

which gives us a monotonically increasing relationship between the birth rate b and the constant rate of population growth g.

Now we consider the effect of the fertility shock. Suppose (for simplicity) that the population was initially growing at the high constant growth rate associated with $b = b^H$, given by equation 3, the effect of the fertility shock leads to a reduction in birth rates described by

$$b_t = \begin{cases} b^H, & t < t^* \\ b^L & t \geq t^* \end{cases}, where\ b^H > b^L. \tag{4}$$

Using equation 2, and noting that the initial demographic state was that of constant growth, we may generate the implied demographic transition thereafter. Iterating forwards in time, we find that the growth rate of the population tends towards the new constant growth rate given by equation 3 with $b = b^L$. Hence the effect of the fertility shock in the medium run is a transition from the initial state of constant high population growth, to the long-run eventual state of low constant growth.

2.1.1 Calibration to Germany

We apply our demographic model to Germany in 2016. Taking the fertile sub-population to be those between the ages of 20 to 30, we begin under the assumption that the population was initially growing at the constant rate consistent with a TFR of 2.5. We model the fertility shock as a sudden drop in TFR from 2.5 to 1.4 in the year of 1970 (see figure 2). Figures 2 and 3 below show the fit of our demographic model.

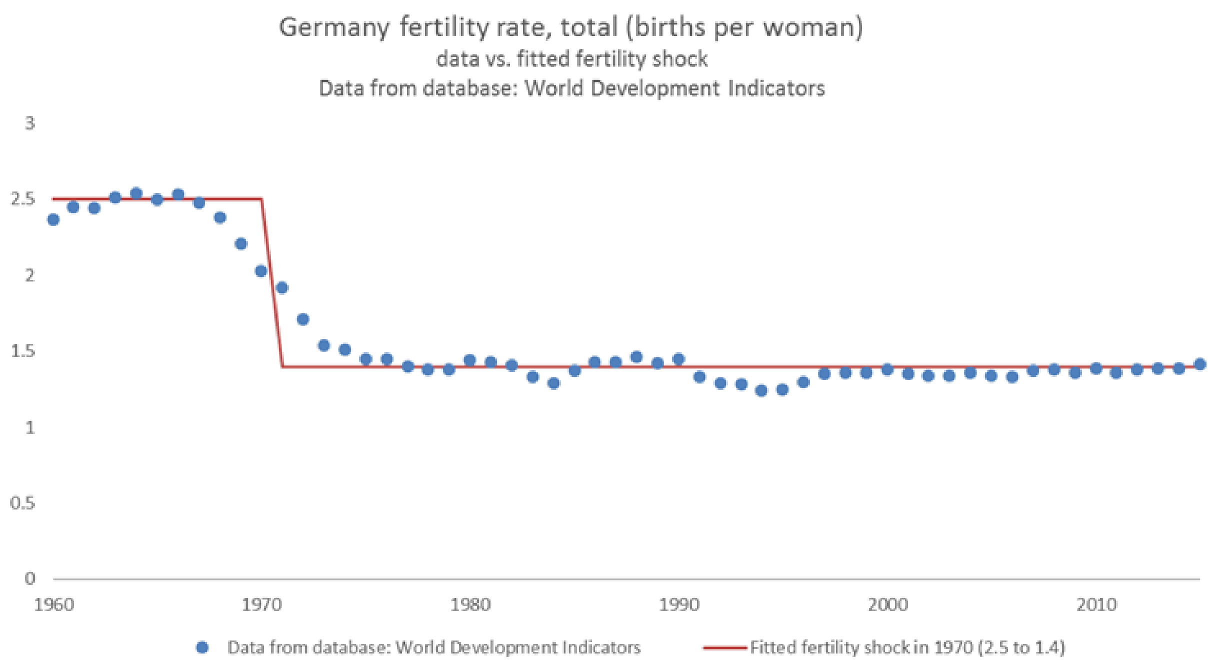

Figure 2: Germany's TFR following the fertility shock

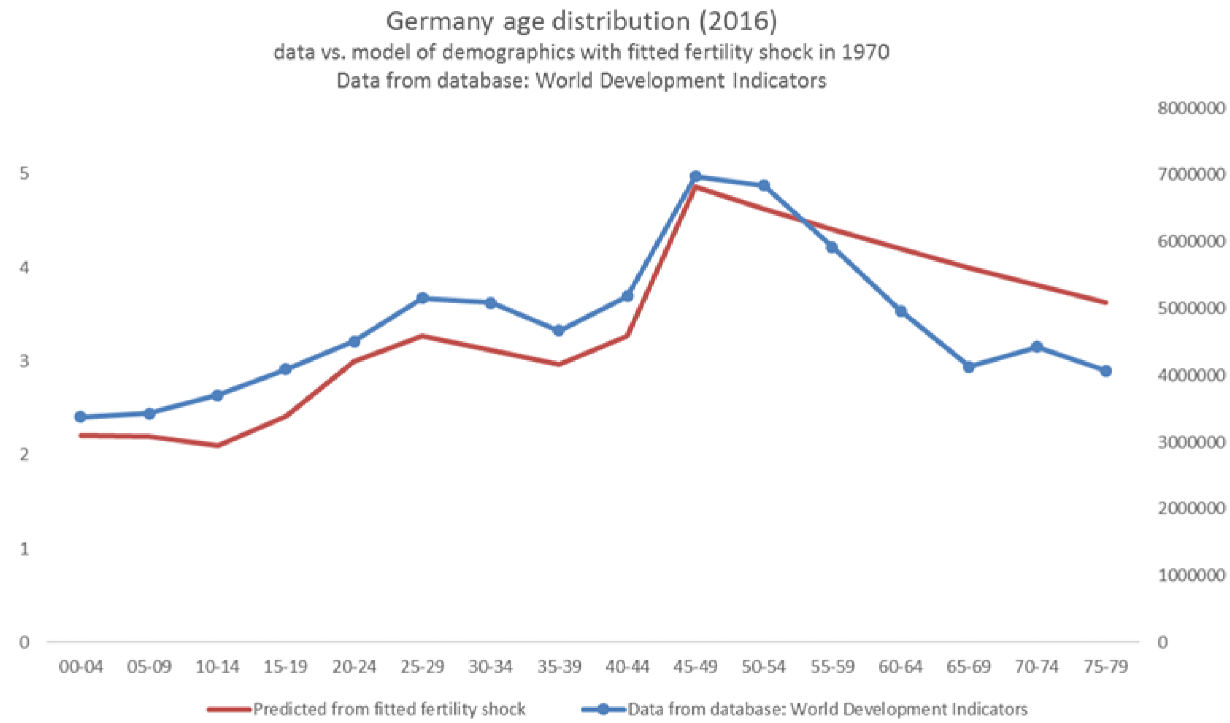

Figure 3: Germany's age distribution in 2016.

Looking at figure 2, we see that our assumption of a sudden fertility shock matches well the decline in fertility rates measured for the German population. In addition, from figure 3 we see that shock can accurately explain the German age distribution in 2016 using our simple demographic model.

The important point to note is that the current age structure for Germany, in terms of its ratio between those approaching retirement to those who are young, is more extreme than the long-run state of constant rate population decay. There is considerably less mass in the young cohorts' part of the Germany's age distribution. This highlights the relative size of the large cohort born before the fertility shock, and the relative scarcity of the absorbers of their savings (the younger cohorts). The implication is that Germany's current demographic structure is intensely biased towards saving. Our hypothesis in this paper is that this transitional phenomenon has profound macroeconomic implications.

We shall take Germany as a point of departure, although Germany is not the only country facing this age structure. Our discussion so far has focused on the four largest economies, however the graphs for Spain, Italy, The Netherlands, and Belgium look similar, somewhat more extreme for Spain and Italy, somewhat less extreme for The Netherlands and Belgium. Only France has a substantially different structure, similar to the pillar shape observed for the United States. To the extent that the Euro-zone has a closed capital market, our analysis therefore applies to the Euro-zone as a whole. The previous discussion suggests that this demographic pattern holds for Japan and China too, but with two differences, one in timing and another in magnitude. Regarding timing: Japan runs 15 years ahead of Germany (and the rest of Euro-zone). China, for that matter, lags 5 years behind the Euro-zone. Regarding magnitude: the magnitude of the distortion to age structure, in Germany and the Euro-zone, dwarfs that of Japan.

3 Stylized model

In this section we the most stripped-down model that demonstrates still the main mechanisms of our paper. Our model considers households that live for two periods. In the first period of their lives the households work and earn an income, some of which they must look to save in order to finance consumption in the second period of their lives (in retirement). The fertility shock leads to a disproportionately large cohort born just before 1970, which we model in our two-generation economy as a one-period positive shock to cohort size.

A household born in period τ has a CES utility function

$$U_\tau = \begin{cases} \frac{{c_{\tau,0}}^{1-\theta}}{1-\theta} + \beta \frac{{c_{\tau,1}}^{1-\theta}}{1-\theta} & for\ \theta \neq 1 \\ log(c_{\tau,0}) + \beta log(c_{\tau,1}) & for\ \theta = 1 \end{cases},$$

where $c_{\tau,i}$ denotes the consumption of the cohort born in period τ at age i, and $1/\theta$ measures the elasticity of intertemporal substitution. This leads to the following Euler equation

$$c_{\tau,1} = \beta^{1/\theta}(1 + r_{\tau+1})^{1/\theta} c_{\tau,0}. \tag{5}$$

There is a Pay As You Go (PAYG) pension system that transfers a proportion γ of the labor income from the young directly to the old generation each period. The household budget constraint for the household born in period τ is given by

$$c_{\tau,0} + c_{\tau,1}(1 + r_{\tau+1})^{-1} \leq w_\tau(1 - \gamma) + \gamma w_{\tau+1} \frac{N_{\tau+1}}{N_\tau}(1 + r_{\tau+1})^{-1}. \tag{6}$$

The aggregate resource constraint is given by

$$Y_t = c_{t-1,1}N_{t-1} + c_{t,0}N_t + I_t, \tag{7}$$

where N_t is the size of the cohort born in period t, and I_t is the level of capital investment in period t.

We assume capital depreciates fully between each period, hence

$$K_{t+1} = I_t. \tag{8}$$

Firms are perfectly competitive and they produce with respect to a Cobb-Douglas production function using capital and labor as inputs:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \tag{9}$$

Labor is supplied inelastically by the young generation in each period, whom are endowed with one unit of labor each:

$$L_t = N_t. \tag{10}$$

Perfect competition in the capital market gives

$$k_t = \left(\frac{\alpha}{r_t + 1} \right)^{\frac{1}{1-\alpha}}, \tag{11}$$

where $k_t \equiv K_t / L_t$ is the level of capital per worker.

Perfect competition in the labor market gives wage as a function of capital per worker

$$w_t = (1 - \alpha) k_t^{\alpha}, \tag{12}$$

which is implicitly a function of r_{t-1} according to equation 11.

An equilibrium in this economy is defined as a path for the following quantities $\{c_{t,0}, c_{t,1}, r_t, w_t, k_t, L_t, K_t, I_t, Y_t\}$, given an exogenous path for N_t and given parameters $\{\alpha, \beta, \gamma, \theta\}$, so that equations 5, 6, 7, 8, 9, 10, 11 and 12 are satisfied in all periods.

Furthermore, a steady state equilibrium is a special case of an equilibrium, as defined above, where N_t grows at a constant rate g, and where the real interest rate and hence all per capita variables are constant.

Next we look to analyze the implications of the fertility shock and the resulting demographic trends common to all advanced economies, within the framework of our stylized model. Note however that in our stylized two generation model it is not sufficient to consider a sudden drop in the birth rates. Consider for example the following demographic model: $N_t = b_t N_{t-1}$. Because households live only for two periods, a drop in the birth rate leads to the immediate adjustment of cohort size growth to the new lower level. This misses the entirety of the demographic transition to the new lower growth state, namely the presence of the large cohort born before the shock. Note that this is not to say that the cohort born before the shock is not "larger": assuming b_t fell from $b^H > 1$ to $b^L < 1$ the cohort born before the shock is still larger than those born before or after. However they are not "large" in the sense that this cohort is exactly the size you would expect them to be, given the size of the next cohort and the rate of population decay. The presence of the large cohort is important as otherwise the transition to the new growth rate would be a simple gradual convergence process, and it is exactly the perverse demographic transition that drives the magnitude of the response for the real interest rate. For this reason we consider the alternative demographic trajectory to understand the implications of this large cohort.

Suppose initially that cohort size is at a constant level of N^L which we normalize to 1. At time t^* there is sudden unexpected one-period increase in cohort size to $N^H = 1.5$. After t^* cohort size returns to the original constant level of N^L. This is summarized by

$$N_t = \begin{cases} N^H = 1.5 & t = t^* \\ N^L = 1 & t \neq t^*. \end{cases} \tag{13}$$

The cohort of size N^H is the large cohort born before the fertility shock. The cohorts born before or after are smaller by a factor of 1/3, which matches approximately the relative sizes of cohorts we see in many countries. For simplicity, we consider constant population both before and after the shock. This corresponds to a population growth rate of 0. Furthermore suppose that the economy was initially operating at the steady state equilibrium consistent with a constant population prior to the fertility sock.

Because the increase in cohort size is known to be transitory to the large cohort, they are aware that the supply of savings they generate is large relative to what the smaller future generations can absorb. Figure 4 shows the transition path of equilibrium real interest rates following the demographic shock described by 13.

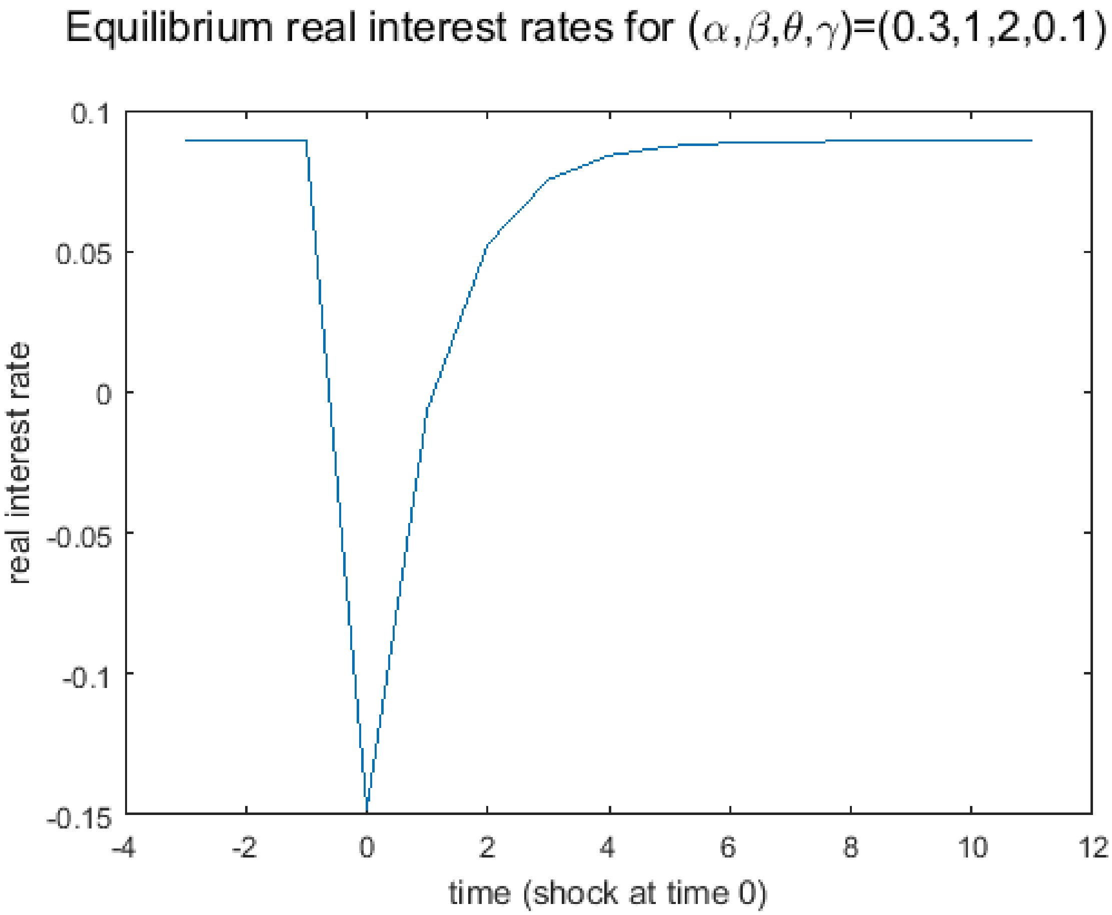

Figure 4: Equilibrium real interest rates for $(\alpha, \beta, \theta, \gamma) = (0.3, 1, 2, 0.1)$.

For our parameter specification, in particular with $\gamma = 0.1$, the steady state real interest rate is just below 0.1, exceeding the growth rate of the economy at $g = 0$. Hence the steady state equilibrium in this economy is dynamically efficient. The results show that real interest rates fall dramatically in period t^*, upon the entry of the large cohort. As time goes on, the fall in real interest rates dies out and tends towards the steady state level. The real interest rate decline is explained by the savings of the large cohort. Due to their relative size, to finance their retirement consumption they save a relatively large amount. As these savings are directed to capital, the return on capital investment falls, leading to also a corresponding fall in real interest rates in equilibrium. After period t^*, the "excess" capital from the large cohort dissipates gradually, generating some persistence in the real interest rate process. Note that this persistence is despite full capital depreciation. Instead the persistence comes from the relatively higher level of capital per worker next period, which leads to relatively higher wages and relatively higher savings. α measures the degree to which capital can absorb savings. For a higher α, the decline in real interest rates is smaller. $1/\theta$ measures the degree to which households are willing to substitute consumption across periods. For a higher θ, the decline in real interest rates is larger.

Regarding timing, the negative response of real interest rates corresponds to the years that the large cohort is active on the labor market and saving for retirement. As the large cohort is born just before 1970, period t^* maps to a decrease in real interest rates from around 1985 to 2035. This fits broadly the experience among the advanced economies. The exact timing and magnitude of this decline shall be examined in closer detail in the next section.

4 Full model

For a realistic simulation of the effects resulting from demographic change, described in section 2, we set the demographic model of section 2 in a general equilibrium environment and calculate the resulting equilibrium real interest rate path.

Households live for a total of J periods (years). Following Barro and Becker (1989) and Curtis et al. (2015), we consider each household to be headed by a singular adult, whom exhibits parental altruism. Due to this altruism, a parent has incentive to provide for his/her dependent children. Before a child reaches adulthood they make no economic decisions, except for the indirect effect they have on their parent's expenditure. After a child reaches adult age and enters the labor market, they become independent and leave the care of their parent(s). A specification of this kind generates a "hump-shaped" consumption pattern for households over their life cycles, which we observe empirically (see Attanasio et al. (1999)).

The utility function for a household for born in period τ is of the following form

$$U_\tau = \begin{cases} \sum_{i=\chi}^{J-1} \beta^i \left(\mu (n_{\tau,i})^\eta \frac{(c^c_{\tau,i})^{1-\theta}}{1-\theta} + \frac{(c^a_{\tau,i})^{1-\theta}}{1-\theta} \right) & for\ \theta \neq 1 \\ \sum_{i=\chi}^{J-1} \beta^i \left(\mu (n_{\tau,i})^\eta log(c^c_{\tau,i}) + log(c^a_{\tau,i}) \right) & for\ \theta = 1 \end{cases},$$

where $c^c_{\tau,i}$ and $c^a_{\tau,i}$ denote the consumptions of the household born in period τ at age i, dedicated to each of his/her children and to himself/herself respectively, η measures the elasticity of a single child's consumption with respect to the number of children, μ measures the overall weighting on children's consumption, and finally $n_{\tau,i}$ measures the number of dependent children a household born in period τ has at age i.

The number of dependent children for a household born in period τ at age i is given by

$$n_{\tau,i} = \sum_{j=\chi}^{min\{\overline{F},i\}} b_{\tau,i} - \sum_{j=\chi}^{min\{\overline{F},i-\chi\}} b_{\tau,i}. \tag{14}$$

Optimal distribution of per-period expenditure across children's consumption versus the adult's consumption gives

$$\frac{c^c_{\tau,i}}{c^a_{\tau,i}} = (\mu n_{\tau,i}^{\eta-1})^{1/\theta}.$$

Using the above optimality relation, we may rewrite the original utility function in terms of overall household consumption

$$U_\tau = \begin{cases} \sum_{i=\chi}^{J-1} \beta^i \frac{(c^a_{\tau,i})^{1-\theta}}{1-\theta}(1+\mu^{1/\theta} n_{\tau,i}^{\eta/\theta-1/\theta+1}) & for\ \theta \neq 1 \\ \sum_{i=\chi}^{J-1} \beta^i log(c^a_{\tau,i})(1+\mu^{1/\theta} n_{\tau,i}^{\eta/\theta-1/\theta+1}) & for\ \theta = 1 \end{cases}, \tag{15}$$

where $c_{\tau,i}$ is the household's overall consumption, defined as $c_{\tau,i} \equiv c^c_{\tau,i} n_{\tau,i} + c^a_{\tau,i}$.

The rewritten utility function allows us to derive the new Euler equation. Supposing that households may borrow or save at the market real interest rate, the utility function given by equation 15 generates the following Euler equation

$$c_{\tau,i+1} = \beta^{1/\theta}(1+r_{\tau+i+1})^{1/\theta} \left(\frac{1+\mu^{1/\theta} n_{\tau,i+1}^{\eta/\theta-1/\theta+1}}{1+\mu^{1/\theta} n_{\tau,i}^{\eta/\theta-1/\theta+1}} \right)^{1/\theta} c_{\tau,i}, \tag{16}$$

which differs from the Euler equation in section 3 due to the new demographic term.

Each household receives a unitary labor endowment each year it is active on the labor market, which begins at age χ and ends at age ψ. Households derive no utility from leisure, and hence they inelastically supply their labor to firms at the going market wage. A household born in period τ has the following budget constraint

$$\sum_{i=\chi}^{J-1} c_{\tau,i} \prod_{j=0}^{i-1} (1+r_{\tau+j+1})^{-1} \leq \sum_{i=\chi}^{\psi-1} w_{\tau+i} \prod_{j=0}^{i-1} (1+r_{\tau+j+1})^{-1}, \tag{17}$$

where $w_{\tau+i}$ is the wage at period $\tau + i$.

Furthermore the aggregate labor supply is given by

$$L_t = \sum_{i=\chi}^{\psi-1} N_{t-i}, \tag{18}$$

where L_t is the aggregate labor supply.

Firms produce subject to a CES production function using two inputs, capital and labor:

$$Y_t = \begin{cases} \left(\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) L_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & for\ \sigma \neq 1 \\ K_t^{\alpha} L_t^{1-\alpha} & for\ \sigma = 1 \end{cases}, \tag{19}$$

where σ measures the elasticity of capital labor substitution, and the limiting case of σ tending to 1 gives the Cobb-Douglas production function. Furthermore the limiting case of α equal to 0 gives the endowment income special case.

Capital depreciates at rate $\delta \in (0,1]$. The equation for capital accumulation is given by

$$K_{t+1} = (1-\delta) K_t + I_t. \tag{20}$$

The aggregate resource constraint is given by

$$Y_t = I_t + \sum_{i=\chi}^{J-1} c_{t-i,i} N_{t-i}. \tag{21}$$

Perfect competition in the capital rental market gives

$$r_t + \delta = \begin{cases} \alpha k_t^{-\frac{1}{\sigma}} \left(\alpha k_t^{\frac{\sigma-1}{\sigma}} + 1 - \alpha\right)^{\frac{1}{\sigma-1}} & for\ \sigma \neq 1 \\ \alpha k_t^{\alpha-1} & for\ \sigma = 1 \end{cases}, \tag{22}$$

and perfect competition in the labor market gives

$$w_t = \begin{cases} (1-\alpha) \left(\alpha k_t^{\frac{\sigma-1}{\sigma}} + 1 - \alpha\right)^{\frac{1}{\sigma-1}} & for\ \sigma \neq 1 \\ (1-\alpha) k_t^{\alpha} & for\ \sigma = 1 \end{cases}, \tag{23}$$

where k_t denotes the level of capital per worker.

An equilibrium in this economy is defined as a path for the following quantities $\{c_{t,\chi}, ..., c_{t,J-1}, r_t, w_t, k_t, L_t, K_t, I_t, Y_t\}$, given an exogenous path for $\{b_t, n_{t,\chi}, ..., n_{t,J-1}, N_t\}$ that satisfies equations 2 and 14, and given parameters $\{\alpha, \beta, \gamma, \eta, \mu, \theta\}$, so that equations 16, 17, 18, 19, 20, 21, 22 and 23 are satisfied in all periods.

Furthermore, a BGP equilibrium is a special case of an equilibrium, as defined above, where N_t grows at a constant rate g, and where the real interest rate and hence all per capita variables are constant.

To characterize the BGP equilibrium, it suffices to find the BGP real interest rate. To do this, we first solve the household and firm problems as a function of the unknown real interest rate. Next we find the implied size of the aggregate capital stock by aggregating the individual households' savings. This corresponds to the demand for assets as function of the real interest rate. The implied size of the aggregate capital stock must also be consistent with the level of capital per worker consistent with the real interest rate. This corresponds to the supply of assets as a function of the real interest rate. Combining these two relations determines the BGP real interest rate uniquely. We derive analytically the equation determining the BGP real interest rate in the appendix.

Applying our calibration to Germany from section 2, we begin with the assumption that the population was initially growing at the high constant rate consistent with $b = b^H$. Although this is a simplifying assumption, we saw from section 2 that the age composition of Germany in 2016 was accurately captured despite this. Suppose further that the economy was initially operating at the BGP equilibrium until the moment of the shock, and that households did not expect the shock prior to the shock. Upon the shock, households need to reevaluate their optimal consumption/saving decisions in response to the new information. We consider a perfect foresight equilibrium thereafter, where households correctly evaluate the future trajectories for economic variables and optimize with respect to this. To calculate the economic transition following the shock, we employ a computational algorithm that iteratively calculates the real interest rate path following the shock. Upon convergence of the iterative procedure, we stop our search and check directly the market-clearing conditions. We describe our iterative algorithm in more detail in the appendix.

Below we show our results for the baseline specification for Germany. Following section 2 we take J, χ, ψ, $\underline{F}$ and $\overline{F}$ to be 75, 20, 65, 20 and 30 respectively. Furthermore we take a standard parametrization for the household and firm parameters, with $\alpha = 0.5$, $\beta = 0.99$, $\delta = 0.1$, $\eta = 0.76$, $\mu = 0.65$, $\theta = 2$ (i.e. an elasticity of intertemporal substitution of 0.5) and $\sigma = 0.4$.[2]

[2] See Chirinko (2004), Havránek (2015), Curtis et al. (2015) and Nadiri (1996) for the rationale of the baseline specification parameters.

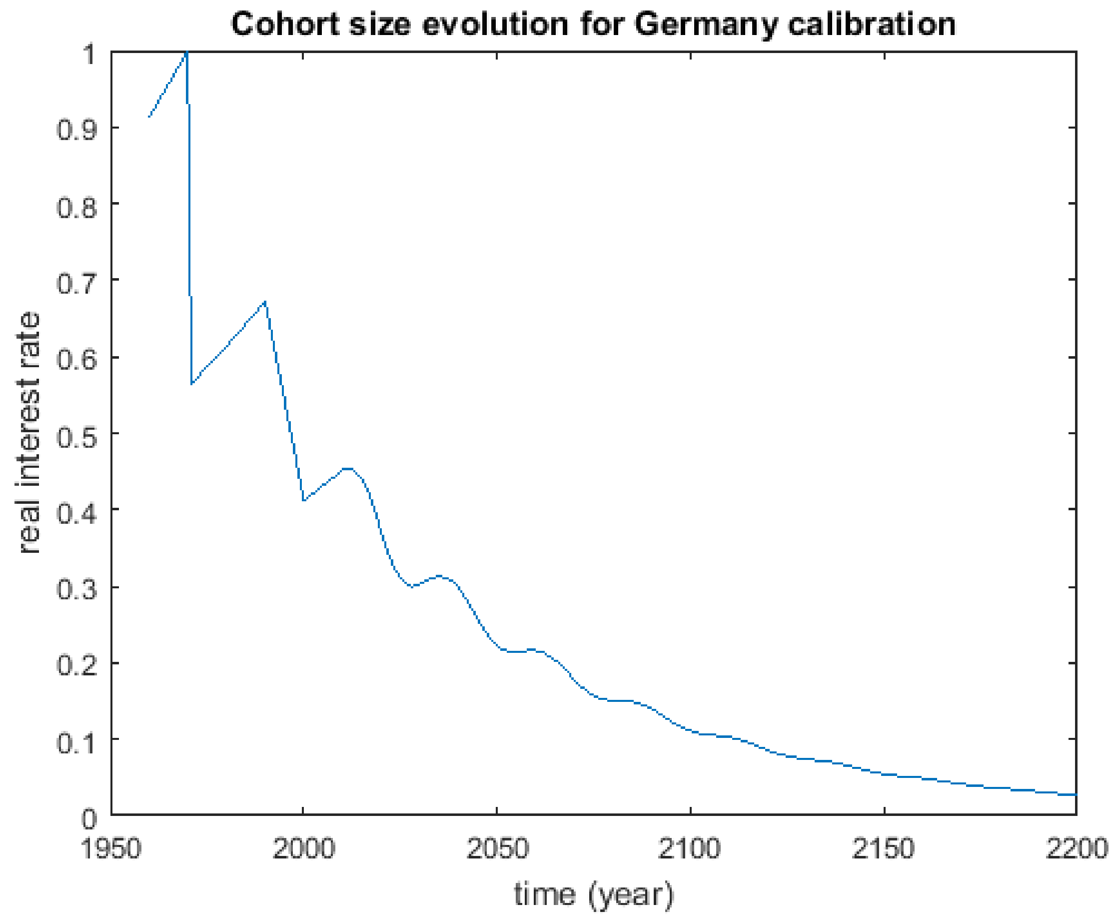

Figure 5: The demographic effect of the pill over a long horizon.

Figure 5 shows the evolution of cohort size after a negative fertility shock in 1970, over a very long horizon. Before the shock, population was growing at a constant rate of 1%. The growth rate shall eventually converge to a constant rate of decline at 1.5% per annum, however in the process of approaching the future long-run growth rate, demography undergoes a transition period of many decades. Following the shock there is an immediate decline in cohort size. This is followed by a short period of gradual increase, as the size of the fertile sub-population is still increasing. This gradual increase is exhausted after 20 years, and following from this the first smaller cohorts begin entering the fertile sub-population. This leads to an echo effect of the original fertility shock and hence cohort size falls sharply again. There are additional echos of the original shock, which die out over time as the age distribution tends to the long-run stable distribution. The periodicity of the echos is 25 years, which corresponds to the average age of a mother.

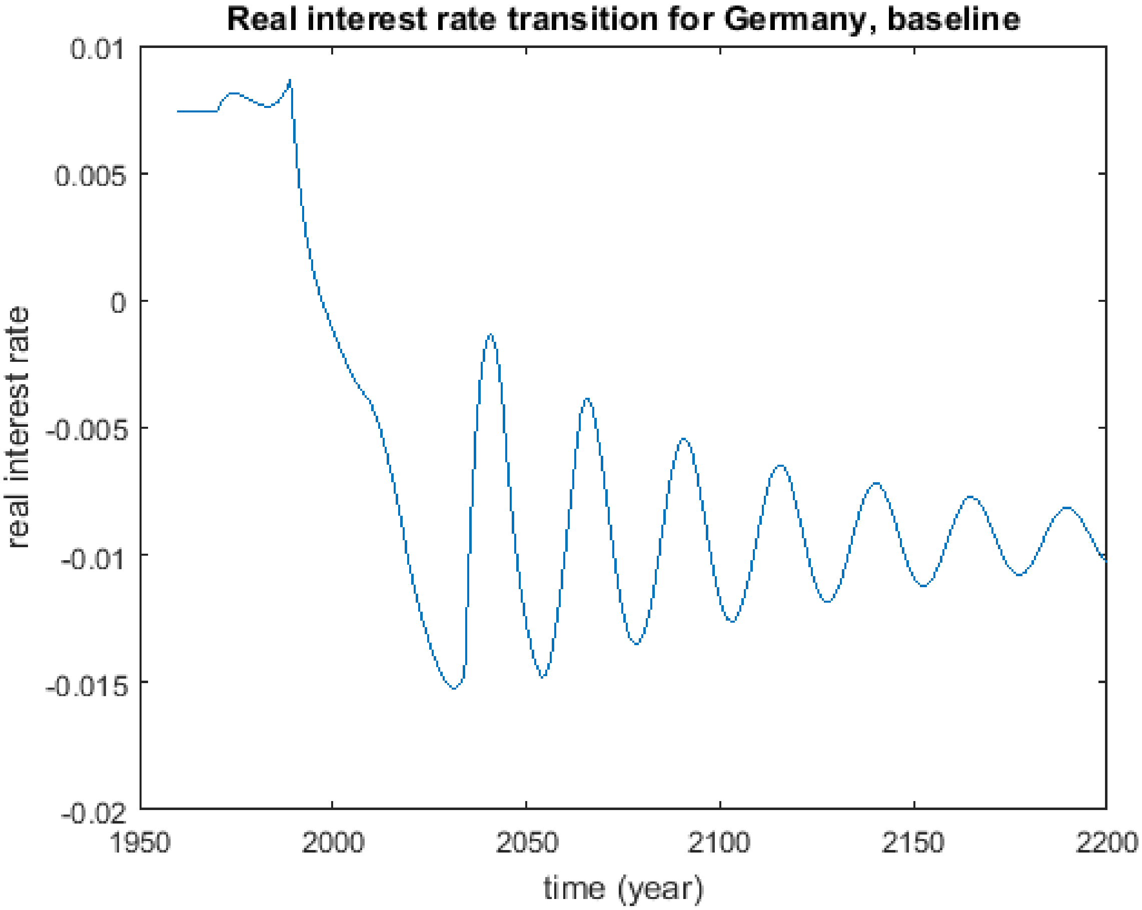

Figure 6: The evolution of the real interest rate over a long horizon.

Figure 6 shows the trajectory of the equilibrium real interest rate for Germany under the baseline parameter specification. Initially the real interest rate moves up. The composition of the population is hardly affected at that time since the number of post-shock cohorts is still low, however the older cohorts feel richer because they realize that in subsequent years the size of future cohorts will be smaller, relaxing the tension on the goods market and making their claim of future output more valuable. Hence, they increase consumption, which drives up the real interest rate. From about 1985 onwards, the large cohort enters the labor market and consequently the real interest rate starts declining steeply, from nearly 1% to -1.5% in 2035, a decline of some 2.5% crossing 0% around the year 2000. What is striking is the length of the period for which the real interest rate declines. The downward pressure continues well into the 2030s when the large cohort begins to retire, and the depth of the trough significantly overshoots below the new BGP level.

After 2035, there is some recovery in real interest rates as the large cohort depletes their savings in retirement. This leads to a reduction in the capital stock and hence an increase in the return of capital investment/saving. The echos of the cohort size distribution, discussed above, generates residual cyclicality in the real interest rate path. Because the first post-shock cohort was much smaller in size than the previous cohorts, when they enter the fertile sub-population their smaller size leads to a reduction in the number of births that year. Subsequent generations of the first post-shock cohorts are likewise also smaller, although the magnitude of this disruption fades out over time. With the birth of each new generation of the original post-pill cohort, demography undergoes a dampened residual shock similar to that of the original fertility shock. This leads to a fall and a subsequent uptick in real interest rates, as is observed in the cyclicality of real interest rates. As noted before the periodicity of the cohort size distribution is 25 years, the average age of a fertile mother. Likewise the periodicity of the real interest rate cycles are also 25 years. As the magnitude of the demographic disruption fades out over time, as does magnitude of the real interest rate cycles. By the year 2250 the transition to the new BGP is more or less complete.

The appendix shows the real interest rate trajectory for some alternative parameter specifications. We offer a brief summary of our comparative exercise here. An increase in life expectancy or a decrease in the retirement age acts to increase the number of years the large cohort needs to finance consumption for in retirement, which leads to a deeper trough in real interest rates. The implication for policy is that raising the retirement age can help raise equilibrium real interest rates. The equilibrium real interest rate in our model, without nominal rigidities, maps to the natural rate of interest in a world with rigidities. Hence increasing the real interest rate in our environment corresponds to an increase in the natural rate of interest, something that may well be desirable when monetary policy is constrained below by the zero lower bound. A decrease in the elasticity of intertemporal substitution deepens the trough in real interest rates, as the elasticity controls the degree to which the large cohort substitutes to saving more when the return on saving falls. Likewise a decrease in the elasticity of capital labor substitution also deepens the trough in real interest rates, as for lower elasticities capital will be able to absorb less savings at a given real interest rate.

5 Conclusion

We show in this paper that the fall in real interest rates, observed in most advanced economies over the past several decades, can be in a large part explained by broad demographic trends resulting from the sudden collapse in fertility rates from the early 1970s. To the degree that these demographic trends are common across the advanced economies, international capital markets offer little help to the problem of low interest rates. Our model predicts that without some reversal in fertility rates, or a dramatic technological breakthrough, low interest rates are here to stay until at least around 2035. In particular Japan's demography provides a rare counterfactual. Because the key features of Japan's demographic profile leads that of the western world by about 15 years, its experience the past two decades provide a forecast of the future for the west. Japan has not escaped from its trap of low interest rates since it entered this period in the early 1990s, and this is precisely what our model predicts given their demography. Besides an increase in the age of retirement, in this paper we can offer little guidance as to how policy can combat the issue of low interest rates. Further research on potential policy responses would be insightful, especially given the problem low interest rates presents for monetary policy at the zero lower bound.

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Appendix

BGP real interest rate

To find the BGP equilibrium real interest rate, we need to find the supply and demand of assets (i.e. capital) as a function of the hypothetical real interest rate. Equating supply and demand and solving for the hypothetical real interest rate gives the BGP equilibrium real interest rate.

Consider a BGP equilibrium. $r_t = r, w_t = w$, and $k_t = k$ and

$$N_t = (1+g)^t,$$

where N_0 is normalized to unity without loss of generality. Define:

$$\begin{aligned} P_i(r) &\equiv (1+r)^{-i} && (24) \\ \Longrightarrow P_i(g) &= N_{-i}, \\ R_i(r) &\equiv \sum_{j=\chi}^{i-1} P_j(r) = \begin{cases} r^{-1}\left(P_{\chi-1}(r) - P_{i-1}(r)\right) & \text{for } r \neq 0 \\ i-\chi & \text{for } r = 0 \end{cases}, \\ W &\equiv w\sum_{i=\chi}^{\psi-1} P_i(r) = wR_\psi(r), && (25) \end{aligned}$$

where W is lifetime wealth of the representative household at birth, and $P_i(r)$ is the intertemporal price of period $t+i$ consumption in terms of period t income, which does not depend on t since $r_t = r$ does not depend on t. Using this notation, the Euler equation given by equation 16 can be written as

$$c_{j+1} = \beta^{1/\theta}(1+r)^{1/\theta}\left(\frac{1+\mu^{1/\theta}n_{j+1}^{\eta/\theta-1/\theta+1}(b)}{1+\mu^{1/\theta}n_j^{\eta/\theta-1/\theta+1}(b)}\right)^{1/\theta} c_j.$$

where c_j is the consumption of the representative household at age j, and $n_i(b)$ is the number of dependent children a representative parent has at age i, given by

$$n_i(b) \equiv \sum_{j=\chi}^{min\{\overline{F},i\}} b - \sum_{j=\chi}^{min\{\overline{F},i-\chi\}} b.$$

Define:

$$g_j^c(b) \equiv \left(\frac{1+\mu^{1/\theta}n_{j+1}^{\eta/\theta-1/\theta+1}(b)}{1+\mu^{1/\theta}n_j^{\eta/\theta-1/\theta+1}(b)}\right)^{1/\theta}.$$

The Euler equation before implies

$$c_j = \beta^{j/\theta} P_j(r)^{-1/\theta} \left({}_{l=\chi}^{j-1} g_l^c(b) \right) c,$$

where c is a constant to be determined.

We next define:

$$\begin{aligned}
B_i(r,b) &\equiv \sum_{j=\chi}^{i} \beta^{j/\theta} P_j(r)^{(\theta-1)/\theta} \left({}_{l=\chi}^{j-1} g_l^c(b) \right), \\
\sum_{j=\chi}^{i} P_j(r) c_j &= \sum_{j=\chi}^{i} P_j(r) \beta^{j/\theta} P_j(r)^{-1/\theta} \left({}_{l=\chi}^{j-1} g_l^c(b) \right) c \\
&= B_i(r,b) c \\
\Longrightarrow W &= \sum_{i=\chi}^{J-1} P_i(r) c_i = B_{J-1}(r,b) c \\
\Longrightarrow c &= \frac{W}{B_{J-1}(r,b)}.
\end{aligned}$$

Let $A_i(r)$ be the stock of assets accumulated by the representative agent at age i. It is equal to the present value of the labor income up to age i, minus the present value of consumption up to that age i:

$$A_i(r) = \sum_{j=\chi}^{i} P_{j-i}(r) \left(w I_{j \in [\chi, \psi-1]} - c_j \right).$$

where $I_x = 1$ if $x = true$ and $I_x = 0$ if $x = false$. This equation is intuitive. The factor $P_{j-i}(r)$ accounts for the discounting of savings and debts. The first term, for the case $i \in [\chi, \psi - 1]$, is the present value of labor income up to age i. The second term is the present value of consumption up to age i.

The exhaustion of the budget constraint gives

$$w \sum_{i=\chi}^{\psi-1} P_i(r) = \sum_{i=\chi}^{J-1} P_i(r) c_i,$$

therefore we have $A_J(r) = 0$, i.e. households spend their full life time wealth on consumption.

We can now calculate the aggregate net demand for assets for all cohorts at period 0, $S(r)$:

$$\begin{aligned} S(r) &= \sum_{i=\chi}^{J-1} N_{-i}A_i(r) = w\sum_{i=\chi}^{\psi-1} N_{-i}\sum_{j=\chi}^{i} P_{j-i}(r) + w\sum_{i=\psi}^{J-1} N_{-i}\sum_{j=\chi}^{\psi-1} P_{j-i}(r) - \sum_{i=\chi}^{J-1} N_{-i}\sum_{j=\chi}^{i} P_{j-i}(r)c_j \\ &= w\sum_{i=\chi}^{\psi-1}\frac{P_i(g)}{P_i(r)}\sum_{j=\chi}^{i} P_j(r) + w\sum_{i=\psi}^{J-1}\frac{P_i(g)}{P_i(r)}\sum_{j=\chi}^{\psi-1} P_j(r) - W\sum_{i=\chi}^{J-1}\frac{B_i(r,b)}{B_{J-1}(r,b)}\frac{P_i(g)}{P_i(r)} \\ &= w\sum_{i=\chi}^{\psi-1}\frac{P_i(g)}{P_i(r)}R_{i+1}(r) + w\sum_{i=\psi}^{J-1}\frac{P_i(g)}{P_i(r)}R_\psi(r) - wR_\psi(r)\sum_{i=\chi}^{J-1}\frac{B_i(r,b)}{B_{J-1}(r,b)}\frac{P_i(g)}{P_i(r)} \end{aligned}$$

Since the capital stock satisfies $K_t = kL_t$, the total net demand for assets minus the next period's capital stock in period 0, $S^*(r)$, is given by

$$S^*(r) = S(r) - K_1 = S - (1+g)\,kL_0.$$

To find the BGP equilibrium real interest rate, it suffices to solve $S^*(r) = 0$. This is given by

$$\sum_{i=\chi}^{\psi-1}\frac{P_i(g)}{P_i(r)}R_{i+1}(r) + \sum_{i=\psi}^{J-1}\frac{P_i(g)}{P_i(r)}R_\psi(r) - R_\psi(r)\sum_{i=\chi}^{J-1}\frac{B_i(r,b)}{B_{J-1}(r,b)}\frac{P_i(g)}{P_i(r)} - (1+g)\frac{k}{w}R_\psi(g) = 0,$$

where $\frac{k}{w}$ is given by

$$\frac{k}{w} = \frac{1}{r+\delta}\frac{\alpha}{(\frac{r+\delta}{\alpha})^{\sigma-1} - \alpha}.$$

Adjustment algorithm

As demography tends from an initial state of high population growth to an eventual state of low population growth, the equilibrium real interest rate must likewise transition from a high level to a low level (the BGP real interest rate is calculated using the equation described above). Note that real interest rates characterize the entire transition, as all allocations may be derived as a function of the real interest rate path (although the allocation in any period shall depend on future interest rates due to forward-looking agents). For the transition between the two BGP's, the real interest rate transition is calculated iteratively using an adjustment mechanism that we describe below.

We assume that the transition period lasts a maximum of $10J$ periods (where J is the life span of a household), i.e. after $10J$ periods the economy has converged to the terminal BGP. For the first iteration, we make the assumption that the real interest rate transition is a linear interpolation between the initial level to the final level. Each successive iteration is then calculated in the following way.

In the $k+1$th iteration of our adjustment algorithm, we begin with the past iteration's real interest rate path, which we denote to be $\{r_t^k\}$ (the initial linear interpolation is denoted by $\{r_t^0\}$), and derive from this the implied path for wages, output, and so on. Using $\{r_t^k\}$ and the implied path of wages, we solve the household problem in *every* period for *every* cohort. This allows us to calculate aggregate consumption demand in each period t:

$$C_t^{k+1} = \sum_{i=0}^{J-1} N_{t-i} c_{t-i,i}^{k+1},$$

where $c_{t-i,i}^{k+1}$ is the demanded consumption in period t by the cohort born in period $t-i$, in iteration $k+1$.

Using the path we calculate for aggregate consumption demand, we can calculate aggregate savings supply by

$$S_t^{k+1} = w_t^{k+1} L_t - C_t^{k+1}.$$

Note the distinction between savings and investment. Savings equals to investment minus capital returns, i.e. savings is the additional quantity devoted to asset accumulation after capital returns is already reinvested. Using savings supply, we can calculate the desired aggregate asset position through time by

$$A_{t+1}^{k+1} = A_t^{k+1}(1 + r_{t+1}^k) + S_t^{k+1},$$

where r_{t+1}^k denotes the real interest rate in period t, iteration k, and the initial asset position is given by $A_{t^*-1}^{k+1} = K_{t^*}$. This allows us to derive the implied capital stock, consistent with the desired asset position, by

$$K_{t+1}^{k+1} = A_t^{k+1}.$$

Note that all assets in our economy our real, i.e. in equilibrium $A_{t+1} = K_t$. In principle there could be financial assets, e.g. money, which would mean that the total amount of assets accumulated by households exceeds the level of the capital stock. Note that the implied capital stock is derived from the *desired* asset accumulation of households. The *actual* quantity of asset accumulated may differ as $1 + r_t^k \neq R_t^K$, the return on capital investment. For the purpose of our algorithm however, we are interested in the *desired* asset position of households, hence we assume that assets grow at the real interest rate of the past iteration, $1 + r_t^k$.

Using the implied capital stock, we may calculate the implied level of capital per worker, and hence induce the implied real interest rate path. We call the path of the implied real interest rates $\{r_t^I\}$. The real interest rate path for the next iteration is finally given by

$$r_t^{k+1} = (1-\phi)r_t^k + \phi r_t^I,$$

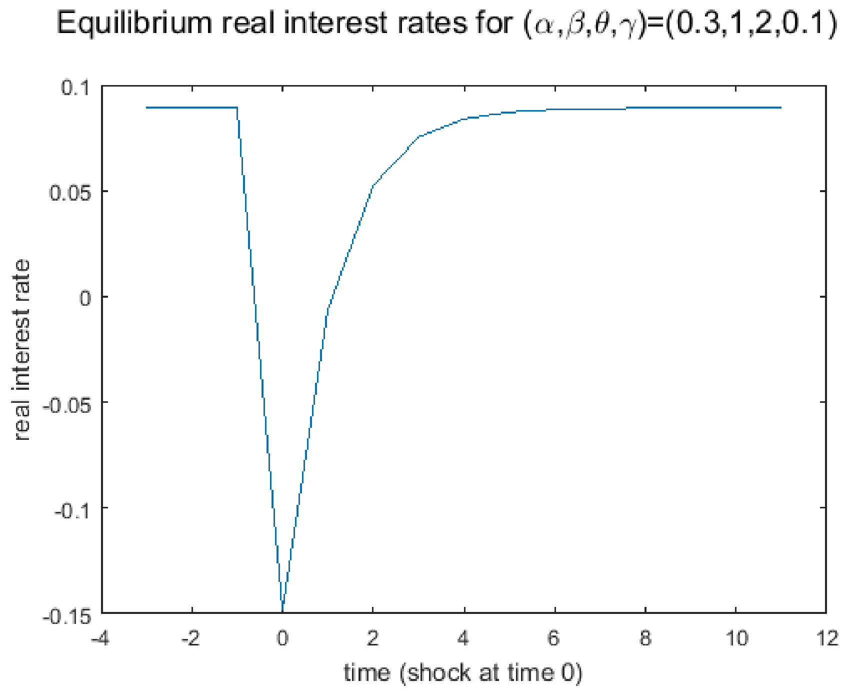
where ϕ is a parameter that controls the degree to which we adjust per iteration. Note that for a stable adjustment, we use a conservative level for ϕ, around the level of 0.05. A larger ϕ leads to over-adjustment and explosive behavior of the algorithm.

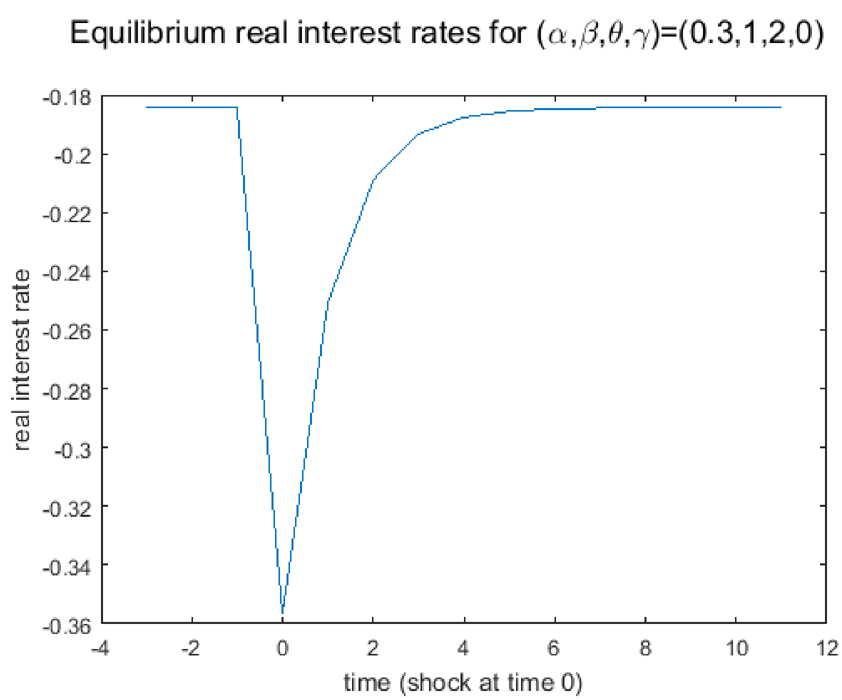
Finally, we compare $\{r_t^k\}$ and $\{r_t^I\}$. If the Euclidean distance between these two vectors is sufficiently small, we have reached convergence and we end our search for the equilibrium transition path at $\{r_t^{k+1}\}$. Note that market clearing is directly guaranteed by convergence, as when $r_t^k = r_t^I$ $\forall t$, it must be the case that in each period aggregate consumption demand equals to aggregate labor income, plus aggregate capital income, minus aggregate capital investment.

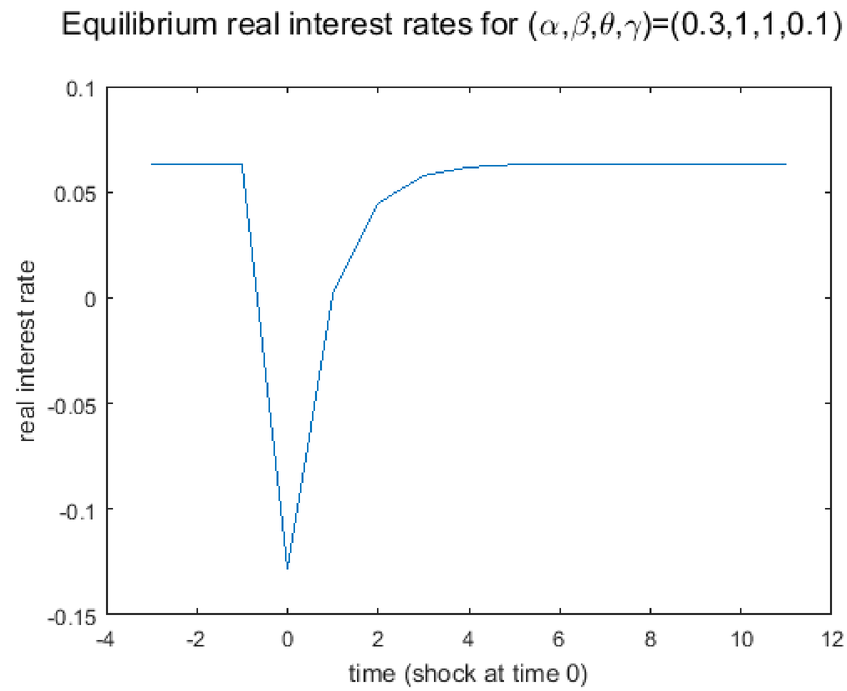
Alternative parameter specifications

We offer alternative parameter specifications for our simulations in sections 3 and 4. In each instance, we shall alter only one parameter, where all remaining parameters are the same as the baseline specification.

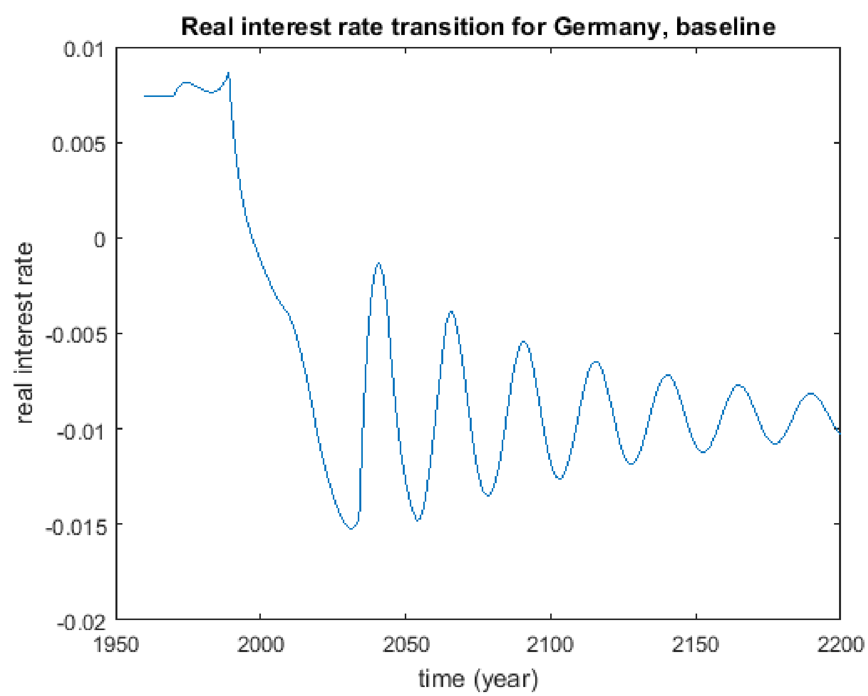
Stylized model

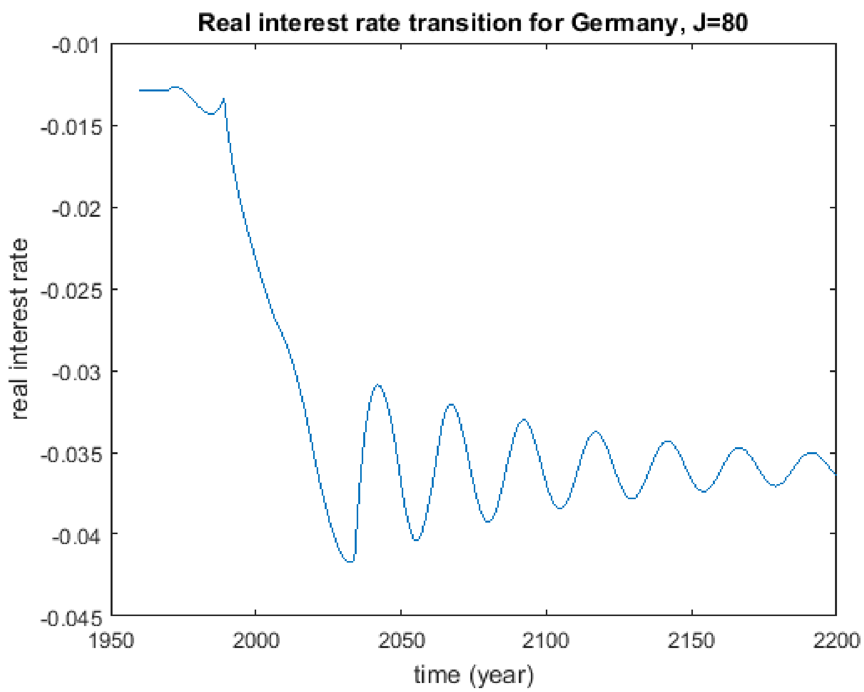




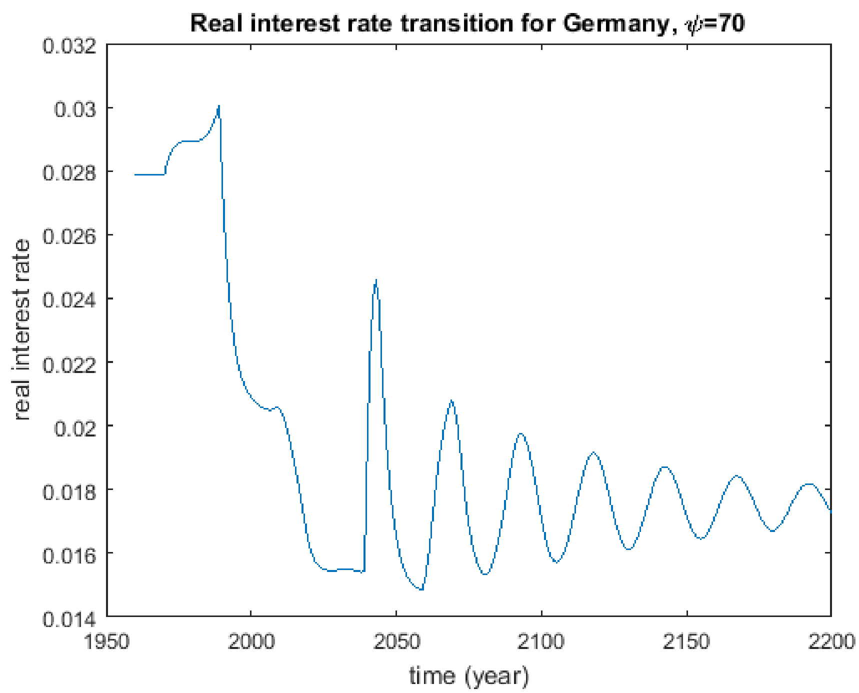

Equilibrium real interest rates for $(\alpha,\beta,\theta,\gamma)$=(0.3,1,1,0.1)
real interest rate
time (shock at time 0)

Full model

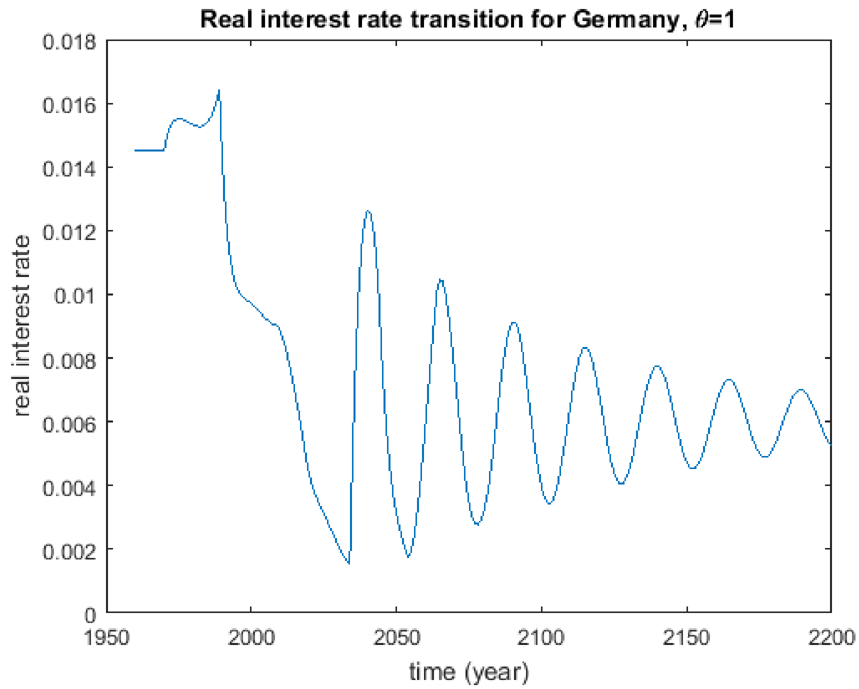

Real interest rate transition for Germany, baseline
real interest rate
time (year)

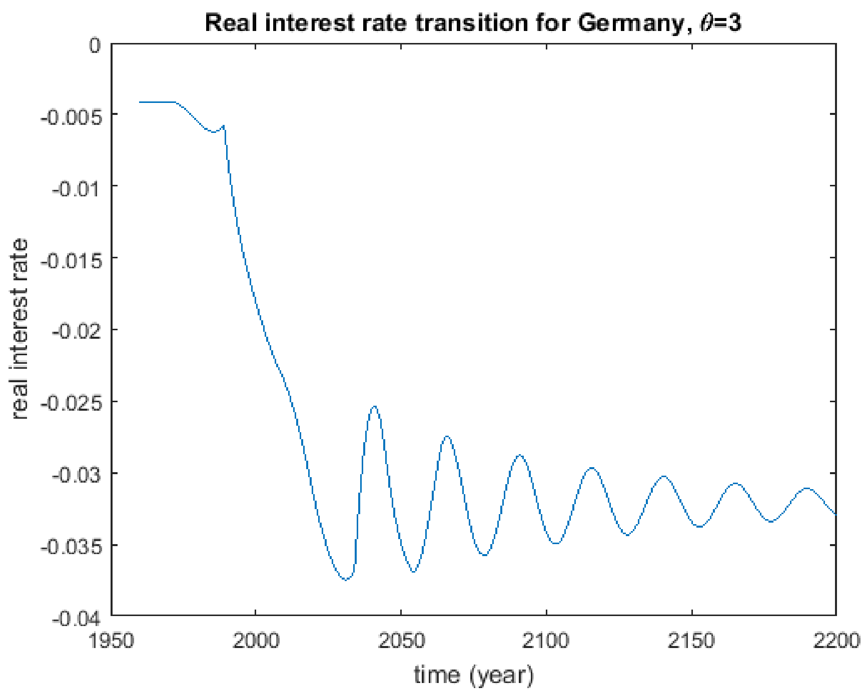


Real interest rate transition for Germany, J=80
real interest rate
-0.01
-0.015
-0.02
-0.025
-0.03
-0.035
-0.04
-0.045
1950
2000
2050
2100
2150
2200
time (year)



Real interest rate transition for Germany, ψ=70
real interest rate
0.032
0.03
0.028
0.026
0.024
0.022
0.02
0.018
0.016
0.014
1950
2000
2050
2100
2150
2200
time (year)


Real interest rate transition for Germany, θ=1
real interest rate
time (year)
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
1950
2000
2050
2100
2150
2200


Real interest rate transition for Germany, θ=3
real interest rate
time (year)
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
-0.035
-0.04
1950
2000
2050
2100
2150
2200