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Asset Prices Under Knightian Uncertainty

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Working Paper No. 172

December 7th, 2021

ABSTRACT

We extend Lucas's classic asset-price model by opening the stochastic process driving dividends to Knightian uncertainty arising from unforeseeable change. Implementing Muth's hypothesis, we represent participants' expectations as being consistent with our model's predictions and formalize their ambiguity-averse decisions with maximization of intertemporal multiple-priors utility. We characterize the asset-price function with a stochastic Euler equation and derive a novel prediction that the relationship between prices and dividends undergoes unforeseeable change. Our approach accords participants' expectations, driven by

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The authors are participants in the Program on Knightian Uncertainty Economics at the Institute for New Economic Thinking (INET). The authors thank Jaroslav Borowicka, Tim Cogley, Halina Frydman, Edmund Phelps, Andrei Shleifer, and Jonathan Stein for insightful reactions and comments on earlier versions of this paper. The authors are grateful to INET for its ongoing support of this research. Johansen gratefully acknowledges support from CREATES - Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation. Rahbek acknowledges funding by The Danish Council for Independent Research (DFF-7015-00028).
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both fundamental and psychological factors, an autonomous role in driving the asset price over time, without presuming that participants are irrational.

https://doi.org/10.36687/inetwp172

JEL Codes: D84, E00, G12, G41.

Keywords: Asset Prices; Unforeseeable Change; Knightian Uncertainty; Muth's Hypothesis; Model Ambiguity; Rational Expectations (REH); Behavioral Finance.

1 Introduction

In a groundbreaking book, Knight (1921) introduced the distinction between "risk" and "true uncertainty." He defined risk as *measurable* uncertainty, which can be represented probabilistically, whereas *true* uncertainty arises from change that cannot "by any method be reduced to an objective, quantitatively determinate probability" *ex ante* (Knight, 1921, pp. 231-232). For Knight, recognizing such change is the key to understanding profit-seeking activity in real-world markets. As he put it: "if all changes (...) could be foreseen for an indefinite period in advance of their occurrence (...) profit or loss would not arise" (Knight, 1921, p. 198).

Today, models formalizing ambiguity about the process driving outcomes are frequently referred to as models with "Knightian uncertainty." However, because such models formalize this ambiguity with a set of stochastic processes that are time-invariant, they ignore Knight's profound insight that true uncertainty arises from *unforeseeable* change. Such change is what makes it inherently impossible to reduce true uncertainty to risk.

Here, we provide a tractable formalization of the Knightian uncertainty faced by an economist and market participants in an intertemporal asset-price model. We do so by opening Lucas's (1978) seminal model to unforeseeable change in the stochastic process for dividends. Adopting a particularly simple formulation, we relate dividends to corporate earnings and assume that the parameter determining the level of earnings, which we refer to as a Knightian uncertainty parameter, undergoes nonrepetitive structural breaks with timings and magnitudes that are not represented with a probabilistic rule, such as a Markov chain.[1] This specification of unforeseeable change in the dividend process formalizes one of the two pillars of our approach: that a model-builder faces Knightian uncertainty about future dividends.

To make our formalization of Knightian uncertainty tractable, we restrict the unforeseeable change that can occur, according to an economist's model. We do so by constraining the Knightian uncertainty parameter to lie within a time-invariant interval at all times. Thus, viewed from any point in time, there are many scenarios for the values that future Knightian uncertainty parameters can take. Because each combination of these parameters indexes a unique conditional density for future dividends, the interval within which the Knightian uncertainty parameter lies defines a set of stochastic processes that characterizes them *ex ante*. According to our model, dividends will subsequently unfold according to one of those

[1] The majority of models assume that the stochastic process driving outcomes does not change over time. When this process is opened to change, it is typically formalized with a stationary Markov chain. Such representations of change are time-invariant in the sense that the parameters of the chain are unchanging. See Hamilton (1988).

stochastic processes.

Our approach assumes ambiguity about the process driving outcomes. However, models assuming such ambiguity typically formalize it with a set of time-invariant stochastic processes. This implies that ambiguity is resolvable: over time, it can be learned which time-invariant processes actually characterize outcomes. By contrast, our formalization of Knightian uncertainty implies that the ambiguity about which process in the set will actually characterize future dividends is unresolvable *ex ante*: given that Knightian uncertainty arises from unforeseeable change, there is no way to determine the values of its *future* parameters even with the infinite sample of historical data. Thus, our formalization of Knightian uncertainty implements Epstein and Schneider's (2007) notion of unresolvable ambiguity in an intertemporal model of aggregate outcomes.

The second pillar of our approach to modeling aggregate outcomes under Knightian uncertainty is the premise that market participants are rational: they are profit-seeking and relate their expectations to some understanding of the process driving outcomes. In order to formalize optimal decisions under Knightian uncertainty, we extend the ambiguity-aversion approach to situations in which the process driving outcomes undergoes unforeseeable change. We represent participants' understanding of this process by implementing Muth's (1961) pathbreaking model-consistency hypothesis when both a model-builder and a representative agent face Knightian uncertainty.

Following Ellsberg's (1961) seminal thought experiments, the literature on ambiguity aversion has shown that decision-making differs when individuals face risk and ambiguity. The prevailing approach to representing decision-making under ambiguity formalizes an agent's preferences with multiple-priors utility, according to which an individual's utility is determined by the minimum expected utility over a set of probability distributions that represents both the presence of ambiguity and the individual's aversion towards it. Gilboa and Schmeidler (1989) originated and axiomatized multiple-priors utility in a static setting, while an intertemporal version was proposed by Epstein and Wang (1994) in the context of an asset-price model.[2] Adopting their formulation, we assume that the consumption and portfolio decisions of a representative agent who faces Knightian uncertainty can be represented with the maximization of intertemporal multiple-priors utility.

Implementing Muth's (1961) hypothesis enables us to formalize the expectations on which the agent's intertemporal utility are based. Muth (p. 315) argued that a "sensible" way for an economist to acknowledge market participants' rationality is to represent their understanding

[2] Epstein and Schneider (2003) axiomatized an intertemporal version of multiple-priors utility. For an overview of models of asset markets assuming ambiguity aversion and further references, see Epstein and Schneider (2010) and Guidolin and Rinaldi (2013).

of the process driving outcomes as being consistent with his own understanding of this process, as formalized by his model.

Muth applied his hypothesis in the context of a model that represents how outcomes unfold over time with a time-invariant stochastic process and called it the rational expectations hypothesis (REH). Muth rendered his representation of market participants' expectations consistent with his own model's predictions by representing them with the conditional expectations of his model's time-invariant stochastic process.

In a model assuming ambiguity about the process driving outcomes, regardless of whether it arises from unforeseeable change, the set of conditional expectations that characterizes future outcomes constitutes the model's prediction of these outcomes. However, we argue that when this ambiguity is formalized with a set of time-invariant stochastic processes, it is incompatible with Muth's hypothesis. In such a case, the hypothesis implies that market participants understand that the ambiguity is resolvable. A model-consistent representation of their expectations must involve a learning mechanism that would resolve the ambiguity asymptotically.

Epstein and Schneider (2007) provide such a mechanism in the context of an agent with intertemporal utility with multiple-priors who initially faces both resolvable and unresolvable ambiguity, where the latter is caused by nonrepetitive change. They show that the learning mechanism reduces the resolvable ambiguity to standard probabilistic risk asymptotically, while the unresolvable ambiguity remains. In this sense, representing *market participants'* ambiguity with time-invariant stochastic processes violates Muth's hypothesis.

In contrast, as Epstein and Schneider (2007) pointed out, there is no learning mechanism that could resolve ambiguity when outcomes undergo unforeseeable change. Thus, opening the model to Knightian uncertainty and the unresolvable ambiguity that such "true uncertainty" engenders on the part of *both* a model-builder and a representative agent has enabled us to rely on Muth's hypothesis to represent participants' expectations.

The constraint that the Knightian uncertainty parameter in the stochastic process for dividends always lies within an interval plays a crucial role in our implementation of Muth's hypothesis. It constrains the set of stochastic processes that characterize future dividends *ex ante*, so that our model's prediction of these outcomes is given by the set of conditional expectations indexed by all combinations of the future Knightian uncertainty parameters within the interval. We argue that *any* of the conditional expectations in this set can serve as a model-consistent representation of the agent's expectations.

To encompass rare and large unforeseeable changes in the dividends process that have led to abrupt booms and busts in the past and might do so in the future, the interval for the Knightian uncertainty parameter must necessarily be wide. Implementing Muth's hypothesis means that market participants understand that the unknown future Knightian uncertainty parameters lie within this wide interval and that it includes extreme changes that occur only rarely. However, as Gajdos *et al.* (2008) and Epstein and Schneider (2010, p. 321) have argued, the set of conditional expectations over which an ambiguity-averse agent maximizes utility need not correspond to the full set of conditional expectations that are logically possible.

Our implementation of Muth's hypothesis formalizes this idea. We represent the agent's expectations in terms of a subset of the stochastic processes that characterize future dividends *ex ante*. To do so, we specify intertemporal utility over a set of conditional expectations indexed by an interval that represents the agent's assessment of the future Knightian uncertainty parameters. We let this interval be an autonomous input to our model. However, to implement Muth's hypothesis, we restrict this interval, at all times, to be a subset of our model's interval for the Knightian uncertainty parameters. In this sense, the set of model-consistent participants' expectations, at each point in time, is autonomous in our model.

For example, after analyzing the available data to assess the recent values of the Knightian uncertainty parameters, the agent might be confident that extreme changes, though possible, will not occur in the near future. Because events in the more distant future are more heavily discounted, they maximize utility over a narrow interval for the future Knightian uncertainty parameters based on their assessment of these parameters' recent values.

We argue that, over time, the agent at least intermittently revises in unforeseeable ways his interval for the future Knightian uncertainty parameters. As new information accrues, the agent assesses whether the Knightian uncertainty parameters have recently changed. When they have, the agent revises his interval for the future Knightian uncertainty parameters, thereby expanding or contracting the set of conditional expectations over which he maximizes intertemporal utility. Moreover, even if the new information leads to the agent's assessment that the Knightian uncertainty parameter has remained unchanged, changes in other factors, such as market sentiment, might lead him to revise his interval, thereby expanding or contracting the set of conditional expectations of future outcomes in unforeseeable ways.

Our formalization of Knightian uncertainty and our implementation of Muth's hypothesis, combined with the assumption that the representative agent maximizes intertemporal utility with multiple-priors, implies that the agent's *optimal* decisions, at each point in time, can be represented as if he maximizes his worst-case expected utility. This worst-case expected utility is unique, as it is based on the conditional expectation of future dividends and asset-prices indexed by the lower bound of the agent's interval for the future Knightian uncertainty parameters. That enables us to use the method in Lucas (1978) to derive, at each point in time, the stochastic Euler equation that characterizes the asset-price in general equilibrium.

Over time, the general equilibrium undergoes unforeseeable change, reflecting unforeseeable change in the parameters in both the dividends process and our model-consistent representation of participants' expectations. However, our formalization allows us to derive the new stochastic Euler equation, indexed by the new parameters, that characterizes the asset price function after such changes. Because our model-consistent representation of the agent's expectations is autonomous and changes in unforeseeable ways over time, the novel prediction of our model is that the asset-price function, which formalizes the *relationship* between the asset price and earnings, undergoes unforeseeable change over time.

Opening the Lucas model to Knightian uncertainty enables the joint formalization of the key insights underpinning the milestone approaches in the development since the 1970s of models of aggregate outcomes resulting from market participants' decisions. These approaches – Phelps' (1970) micro-foundations approach, REH, behavioral-finance models, and the ambiguity-aversion approach – are mutually incompatible, on logical grounds, when the stochastic process driving outcomes is assumed to be time-invariant.[3]

However, introducing Knightian uncertainty into the Lucas model reconciles Muth's hypothesis, which underpins the REH approach, with behavioral-finance models' core premise that non-fundamental factors, such as market sentiment, exert an autonomous influence on participants' expectations. As in REH models, imposing consistency within a model open to unforeseeable change relates participants' expectations of aggregate outcomes to fundamentals. Remarkably, once we recognize Knightian uncertainty in how outcomes unfold over time, Muth's hypothesis also plays a central role in representing the influence of market sentiment on participants' expectations.

In contrast, because they maintain REH's premise that outcomes unfold according to a time-invariant stochastic process, behavioral-finance theorists have had to jettison Muth's hypothesis and represent the influence of non-fundamental factors with model-inconsistent representations of participants' expectations. Such representations presume that participants are grossly irrational, in the sense that they commit systematic forecast errors over an indefinite future.

By relying on Muth's hypothesis and representing optimal decisions by market participants facing Knightian uncertainty with the maximization of intertemporal utility with multiple priors, our approach opens a way to build macroeconomic and finance models that accord participants' expectations an autonomous role in driving aggregate outcomes, without

[3]For a formal demonstration and extensive discussion see Frydman and Goldberg (2007) and Frydman and Phelps (2013).

presuming that participants are irrational. In this sense, our approach enables economists to realize the vision that motivated Phelps' (1970, p. 22) micro-foundations agenda: because market participants "maximize relative to their" own imperfect understanding of how the economy works, their expectations play an autonomous role in driving aggregate outcomes. The REH approach preempted this vision, because imposing consistency seemed to rule out such a role for participants' expectations. However, this has rendered macroeconomics and finance models incompatible with compelling evidence regarding psychological factors' influence on participants' expectations.[4]

The paper is structured as follows. In Section 2, we situate our approach in the context of related literature. Section 3 specifies dividends with a stochastic process that is open to unforeseeable change and Knightian uncertainty arising from such change. In Section 4, we implement Muth's hypothesis under Knightian uncertainty. Section 5 specifies the intertemporal utility with multiple-priors. In Section 6, we derive the stochastic Euler equation characterizing general equilibrium asset prices. Section 7 illustrates formally the autonomous role of expectations and the influence of market sentiment in driving asset prices implied by our opening of the model to Knightian uncertainty, while maintaining Muth's hypothesis. The concluding Section 8 argues that opening intertemporal models to Knightian uncertainty on the part of an economist and market participants offers a way forward for macroeconomics and finance theory.

2 Related Literature

There is a substantial literature introducing ambiguity into economic models. An overview of the many models that have been developed since the reviews by Epstein and Schneider (2010) and Guidolin and Rinaldi (2013) would require a separate paper. However, as we note throughout this paper, this literature typically formalizes ambiguity with a set of time-invariant stochastic processes; thus, it does not relate ambiguity explicitly to unforeseeable change. The only exception that we are aware of is Ilut and Schneider (2014), which we discuss below and formally relate to our approach in Section 7.

Although our approach differs from the literature in introducing unforeseeable change into models of aggregate outcomes, it does rely on a specification of preferences that is widely used in the literature on ambiguity. In this respect, we build on Epstein and Wang (1994), which provides an early extension of Lucas's (1978) model by assuming that an ambiguity-

[4]For early reviews of this evidence see Barberis, Shleifer, and Vishny (1998) and Shleifer (2000). Throughout their book, Gennaioli and Shleifer (2018) discuss subsequent studies documenting the influence of non-fundamental factors on market participants' expectations and asset prices.

averse representative agent makes his consumption and portfolio decisions by maximizing an intertemporal utility function with multiple-priors. They formalize ambiguity with a set of time-invariant stochastic processes and focus on situations where the minimum expected utility over this set, and thus also the derivative of the intertemporal utility function, is not uniquely determined by one of the probability distributions in the set. They show that in such situations their model implies an indeterminacy: the asset-price function is characterized by a set of stochastic Euler equations.

Although Epstein and Wang (1994, p. 284) mention that ambiguity could arise from change, they do not explicitly relate the representative agent's ambiguous expectations to unforeseeable change in the consumption and dividends processes. However, as Epstein and Schneider (2007) later showed, if dividends were characterized over time by any of the stochastic processes within the set of time-invariant stochastic processes that represent expectations, as Epstein and Wang assume, the ambiguity would be resolvable in the long run. Thus, Muth's hypothesis would imply that in the long run, market participants do not face ambiguity, because it reduces to standard probabilistic risk. In that case, the asset-price indeterminacy arising from ambiguity would also disappear in the long run.

In contrast, in our extension of Lucas's (1978) intertemporal asset-price model, ambiguity does not vanish asymptotically. It is unresolvable. Moreover, the minimum expected utility that determines the intertemporal utility with multiple-priors is uniquely determined by the assumption that the conditional distribution indexed by the future Knightian uncertainty parameters takes the value at the lower bound of the agent's interval. Thus, in contrast to Epstein and Wang (1994), our model implies that the asset-price function is characterized by a unique stochastic Euler equation *at each point in time.* More importantly, our model implies that the stochastic Euler equation, and thus the asset-price function, changes over time in unforeseeable ways. Consequently, our model recognizes Knightian uncertainty about future asset prices: the future asset-price *functions* are unknown at all times, although Muth's hypothesis restricts them *ex ante* to lie within an interval.

Our formalization of unforeseeable change and the Knightian uncertainty arising from it is related to the formalization of ambiguity about total factor productivity (TFP) in Ilut and Schneider's (2014) New Keynesian business-cycle model. Ambiguity in their model arises from change in the parameter determining the level of TFP shocks. As in our model, this parameter (which we refer to as a Knightian uncertainty parameter to relate it to our approach) is not characterized with a probabilistic rule.

Ilut and Schneider restrict the Knightian uncertainty parameters by assuming that their sample moments converge toward those of an i.i.d. normally distributed process with mean zero and constant variance – a property they use in the solution of their model. Although the unforeseeable change makes the ambiguity about future TFP shocks unresolvable, this restriction implies that the TFP process resembles a time-invariant stochastic process asymptotically. In this sense, their model is not open to unforeseeable change in the long run.

As we do, Ilut and Schneider consider an ambiguity-averse representative agent with multiple-priors utility and represent the agent's expectations in terms of an interval that reflects his "confidence" about the unknown future Knightian uncertainty parameters. As the agent is ambiguity-averse, his optimal decisions can be represented as if he maximizes utility based on worst-case expectations, with the future Knightian uncertainty parameters taking the value at the lower bound of the agent's interval.

Ilut and Schneider assume that the bounds of the agent's interval evolve according to a time-invariant autoregressive process. In contrast, we argue that a utility-maximizing agent facing Knightian uncertainty, at least intermittently, revises this interval in unforeseeable ways.

Leaving aside technical details of the models' solutions, both approaches imply that the aggregate outcomes, such as GDP or the asset price, undergo unforeseeable change. However, representing the agent's interval for the future Knightian uncertainty parameters with an autoregressive process, as Ilut and Schneider do, implies that the unforeseeable change in the outcome variable arises *solely* from the unforeseeable change in the stochastic process driving the input variable, such as TFP or earnings.

Analogously, we specify the process driving the input variable to undergo unforeseeable change. However, as we formally illustrate in Section 7, the distinctive implication of our approach is that the relationship between the input variable and the outcome, such as GDP or the asset price, also undergoes unforeseeable change. Thus, in contrast to Ilut and Schneider's model, our approach implies that the change in the aggregate outcome arises from two sources: unforeseeable change in the process driving the input variable and unforeseeable change in how the input variable is mapped onto worst-case expectations.

Which of these predictions better characterizes the data needs to be empirically investigated by examining whether the relationships between GDP and TFP or asset prices and earnings undergo structural breaks, and whether these breaks can be represented with probabilistic rules.

Our approach builds on the ideas that motivated Frydman and Goldberg's (2007, 2013) attempt to formulate an approach to macroeconomic theory – which they called Imperfect Knowledge Economics (IKE) – that recognizes the importance of unforeseeable change in the process driving aggregate outcomes. Lacking the appropriate mathematical framework to characterize Knightian uncertainty in this process, Frydman and Goldberg could not rely on Muth's hypothesis to represent participants' expectations. Consequently, they could not develop a coherent approach to building intertemporal models that recognizes that economists as well as market participants face Knightian uncertainty about the process driving outcomes.

3 Unforeseeable Change and Knightian Uncertainty About Future Dividends and Earnings

The starting point of our asset-price model is that the stochastic process driving dividends changes over time. A standard approach would be to represent such change with a probabilistic rule. For example, the asset-price models of Cecchetti, Lam, and Mark (1990) and Timmermann (2001) represent change in the dividends process with Markov switching processes.[5] However, such probabilistic representations of change imply that market participants face only what Knight (1921) called "risk."

Instead, adhering to Knight's original definition, we formalize Knightian uncertainty by opening the stochastic process driving dividends to unforeseeable change. We do so by assuming that one of the parameters of this process undergoes structural change at times and with magnitudes that are unknown *ex ante* even in probabilistic terms. This formalization of unforeseeable change gives rise to Knightian uncertainty, in the sense that at any point in time, the probability distribution that characterizes future outcomes is unknown.

We illustrate our approach with a particularly simple model where dividends d_t depend on corporate earnings x_t and the log of earnings follows an autoregressive process. Specifically, we assume that (d_t, x_t) evolve according to

$$d_t = \beta x_t, \qquad \log x_t = \rho \log x_{t-1} + \mu_t + \varepsilon_t, \tag{1}$$

for $t = 1, 2, \ldots$ and where $\varepsilon_t \sim i.i.d.N\left(0, \sigma^2\right)$, $0 < \beta \leq 1$, $0 < \rho < 1$, and the initial value x_0 is given. The stochastic error term ε_t represents standard probabilistic risk and (β, ρ, σ^2) are standard constant parameters.

In contrast, we leave the model open to unforeseeable change in the parameter μ_t, which we refer to as a Knightian uncertainty parameter. The defining feature of such change is

[5] Cecchetti, Lam, and Mark (1990) consider a general equilibrium asset-pricing model where log dividends are assumed to follow a random walk with a drift term, subject to a two-stage Markov switching process, as introduced by Hamilton (1989). Timmermann (2001) extends their model by allowing for an expanding set of nonrecurring regimes where the switching probabilities and parameters in each regime are drawn from fixed distributions, while Pettenuzzo and Timmermann (2011) consider a similar nonrecurring Markov switching model for predictions of stock returns. Ang and Timmermann (2012) provide an overview of regime switching models in finance.

that it is not characterized by a probabilistic rule. Instead, we formalize this change by assuming, first, that μ_t undergoes structural change at times $\{\tau_j\}_{j=1}^{\infty}$ with

$$\mu_t = \bar{\mu}_j, \qquad \text{for } t = \tau_{j-1}, \tau_{j-1}+1, \ldots, \tau_j - 1 \text{ and } j = 1, 2, \ldots, \tag{2}$$

where $\tau_0 = 1$, $\bar{\mu}_j$ are constant parameters, $\bar{\mu}_j \neq \bar{\mu}_{j-1}$, and $\tau_{j-1} < \tau_j$ for all $j = 1, 2, \ldots$. Second, we assume that future changes are unknown *ex ante*, in the sense that, at time t with $\tau_{j-1} \leq t \leq \tau_j - 1$, the timing of future change $\{\tau_i\}_{i=j}^{\infty}$ and its parameters $\{\bar{\mu}_i\}_{i=j}^{\infty}$ are unknown even in probabilistic terms. Although μ_t changes only intermittently, this specification acknowledges that change is unforeseeable: exactly when and how μ_t changes in the future is unknown *ex ante*. We summarize the foregoing considerations with the following assumption.

Assumption 1 *The future Knightian uncertainty parameters $\{\mu_{t+i}\}_{i=1}^{\infty}$ are unknown at all times t.*

To make our approach tractable, we restrict the values that μ_t can take at all times. We do so by constraining the parameter $\bar{\mu}_j$ in each of the subperiods $j = 1, 2, \ldots$ to lie within a time-invariant interval I^{μ}:

$$\bar{\mu}_j \in I^{\mu} = \left[\mu^L, \mu^U\right], \qquad j = 1, 2, \ldots, \tag{3}$$

where $\mu^L < \mu^U$. The specification in (2) and (3) implies that $\mu_t \in I^{\mu}$ for all t.

Using a statistical method such as the Bai and Perron (1998) test or the step-indicator saturation approach by Castle, Doornik, and Hendry (2012) and Castle *et al.* (2015), the number of breaks that have occured in the past, the breakpoints τ_j, and the parameters $\bar{\mu}_j$ during the different subperiods can be estimated based on a sample of historical data for earnings, $\{x_t\}_{t=1}^{T}$. The crucial implication of unforeseeable change, however, is that the *future* timings and magnitudes of change in μ_t cannot be assessed, even on the basis of an infinite sample of historical data.

The empirical estimates of the past values of μ_t enable an assessment of the width of the interval I^{μ} in (3). Importantly, this interval must necessarily be quite wide to encompass rare and large unforeseeable changes that have led to abrupt booms and busts in dividends and earnings in the past and might occur in the future.

To be sure, our model in (1)-(3) represents how dividends and earnings unfold over time with a stochastic process. Opening this process to unforeseeable change nevertheless gives rise to Knightian uncertainty about future dividends and earnings, as the uncertainty about these outcomes cannot *ex ante* be "reduced to an objectively, quantitatively determinate probability" (Knight, 1921, pp. 231–232). This is because the conditional densities of future dividends and earnings are indexed by the unknown future Knightian uncertainty parameters and thus are inherently unknown *ex ante*.

For example, let $f\left(d_{t+1}|x_t;\mu_{t+1}\right)$ denote the conditional density of d_{t+1} given x_t indexed by the Knightian uncertainty parameter μ_{t+1} (and the time-invariant parameters β, ρ, and σ^2), as implied by (1). At all times t, this density is unknown because μ_{t+1} is unknown. In this sense, allowing for unforeseeable change implies Knightian uncertainty about d_{t+1} at all times t.

We emphasize that the Knightian uncertainty arises from the *ex ante possibility* that unforeseeable change can occur at any future time: there is Knightian uncertainty about d_{t+i}, $i > 0$, at time t even if it subsequently turns out that no unforeseeable change occurs between time t and $t+i$.

3.1 Knightian Uncertainty as Unresolvable Ambiguity

A central feature of our approach is that in (3) we restrict the parameter $\bar{\mu}_j$ to lie within the interval I^μ during all subperiods $j = 1, 2, \ldots$. This implies that $\mu_t \in I^\mu$ at all times t, thereby restricting the extent of unforeseeable change that can occur.

It follows that, viewed from any time t, there are many scenarios for the values that the unknown future Knightian uncertainty parameters can take. These are given by all combinations of $\mu_{t+1}, \mu_{t+2}, \ldots$ within the interval I^μ. Each of these possible combinations indexes a unique conditional density of future dividends d_{t+i}, $i > 0$, given current earnings x_t, which we emphasize by denoting this density with $f\left(d_{t+i}|x_t;\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}\right)$. Viewed from time t, any of these conditional densities might characterize future dividends d_{t+i}. Thus, as the interval I^μ restricts the values the unknown future Knightian uncertainty parameters can take, it defines a set of conditional distributions that characterizes future dividends *ex ante*. As the actual dividends evolve according to (1)-(3), one of these distributions actually represents how dividends subsequently unfold.

For example, viewed from time t, both μ_{t+1} and μ_{t+2} can take any value within I^μ. This implies that d_{t+2} is characterized *ex ante* by the set of conditional densities, which we denote with $\mathcal{F}\left(d_{t+2}|x_t;I^\mu\right)$, given by

$$\mathcal{F}\left(d_{t+2}|x_t;I^\mu\right) = \left\{f\left(d_{t+2}|x_t;\mu_{t+1},\mu_{t+2}\right) \mid \mu_{t+1},\mu_{t+2} \in I^\mu\right\}. \tag{4}$$

Subsequently, the actual dividends d_{t+1} and d_{t+2} evolve according to (1) with, for example, $\mu_{t+1} = \bar{\mu}_j$ and $\mu_{t+2} = \bar{\mu}_{j+1}$.

Our formalization of Knightian uncertainty, like models assuming ambiguity about the process driving outcomes, characterizes future outcomes *ex ante* with a set of conditional distributions. However, because the change cannot be estimated based on observed historical data, the ambiguity about future Knightian uncertainty parameters is inherently unresolvable. This implies that which conditional density within the set will actually characterize future dividends "cannot by any method" be determined *ex ante*. Thus, our formalization of Knightian uncertainty implements Epstein and Schneider's (2007) notion of "unresolvable ambiguity."

Epstein and Schneider (2007) illustrate the difference between resolvable and unresolvable ambiguity using a dynamic extension of Ellsberg's (1961) classic urn experiment that corresponds to a simple, discrete version of our stochastic process for dividends and earnings. Over time, the composition of balls is subject to change that is not characterized with a probabilistic rule. Yet, equivalent to our restriction in (3), the composition is bounded, because the five balls in the urn can be only either white or black. Epstein and Schneider argue that change in the composition of balls gives rise to what they call unresolvable ambiguity. This is exactly what the stochastic process in (1)-(3) implements as it undergoes bounded unforeseeable change.

3.2 The Model's Prediction of Future Dividends

Our model's prediction of future dividends is given by the set of conditional expectations of these outcomes indexed by all combinations of the Knightian uncertainty parameters within the interval I^{μ}.

To specify this formally, let $E\left(d_{t+i}|x_t;\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}\right)$, $i>0$, denote the conditional expectation of d_{t+i}, given x_t, indexed by the Knightian uncertainty parameters $\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}$. The process in (1) implies that this is given by

$$E\left(d_{t+i}|x_t;\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}\right)=\beta\left(x_t\right)^{\rho^i}\exp\left(\sum_{j=1}^{i}\rho^{i-j}\mu_{t+j}\right)\omega_i, \tag{5}$$

where

$$\omega_i=E\left(\exp\left(\sum_{j=1}^{i}\rho^{i-j}\varepsilon_{t+j}\right)\right)=\exp\left(\frac{\sigma^2}{2}\sum_{j=1}^{i}\rho^{2j}\right). \tag{6}$$

While the future Knightian uncertainty parameters, $\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}$, are unknown at time t under Assumption 1, these parameters are constrained to lie within the interval I^{μ}. It follows that our model's prediction of future dividends, d_{t+i}, $i>0$, is given by the set of conditional expectations indexed by all combinations of $\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}$ within I^{μ}. To emphasize that this set depends on the interval I^{μ}, we denote it by $\mathcal{E}\left(d_{t+i}|x_t;I^{\mu}\right)$, and it is defined as

$$\mathcal{E}\left(d_{t+i}|x_t;I^\mu\right)=\left\{E\left(d_{t+i}|x_t;\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}\right)\ |\ \mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}\in I^\mu\right\}, \tag{7}$$

where $E\left(d_{t+i}|x_t;\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}\right)$ is given in (5).

An implication of our model, important for our specification of intertemporal utility, is that the conditional expectation in (5) is monotonically increasing in the parameters $\mu_{t+1},\mu_{t+2},\ldots,\mu_{t+i}$. This implies that the set of conditional expectations $\mathcal{E}\left(d_{t+i}|x_t;I^\mu\right)$ in (7) simplifies to the interval given by

$$\begin{aligned}\mathcal{E}\left(d_{t+i}|x_t;I^\mu\right) &= \left[E\left(d_{t+i}|x_t;\mu^L\right),E\left(d_{t+i}|x_t;\mu^U\right)\right]\\ &= \beta\left(x_t\right)^{\rho^i}\omega_i\left[\exp\left(\sum_{j=1}^{i}\rho^{i-j}\mu^L\right),\exp\left(\sum_{j=1}^{i}\rho^{i-j}\mu^U\right)\right],\end{aligned} \tag{8}$$

where we use the notation $E\left(d_{t+i}|x_t;\mu^L\right)$ to denote the conditional expectation of d_{t+i}, given x_t, indexed by $\mu_{t+1}=\mu_{t+2}=\ldots=\mu_{t+i}=\mu^L$.

This shows that, at each time t, the bounds of the set $\mathcal{E}\left(d_{t+i}|x_t;I^\mu\right)$ are uniquely determined: they are indexed by the future Knightian uncertainty parameters taking the values μ^L and μ^U for $i=1,2,\ldots$. Viewed from any point in time t, the set of conditional expectations $\mathcal{E}\left(d_{t+i}|x_t;I^\mu\right)$ in (8) constitutes our model's prediction of future dividends d_{t+i}, $i>0$.

4 Muth's Hypothesis Under Knightian Uncertainty

According to the Merriam-Webster Dictionary, the words "rational" and "reasonable" are synonyms. Both imply "a latent or active power to make logical inferences and draw conclusions that enable one to *understand* the world (...) and relate such knowledge to the attainment of ends" (emphasis added).

Thus, in order to base the micro-founded analysis of aggregate outcomes on market participants' rationality, an economist must represent how they understand "the way the economy works" (Muth, 1961, p. 315). It is self-evident that there is a diversity of ways to understand the economy's structure and how it evolves over time. In proposing how economists could acknowledge market participants' rationality, Muth (1961) appealed to the core premise of all economic models: that modeling payoff-relevant outcomes formalizes an economist's reasonable (theoretically- and empirically-based) understanding of the actual ("objective") uncertainty about them. We paraphrase his striking hypothesis as follows:

> An economist can relate market participants' expectations to rational considerations by specifying their understanding of the process driving outcomes as being consistent with the economist's own understanding of this process, as formalized by his model.

Muth implemented his hypothesis in a model that assumed that outcomes are driven by a time-invariant stochastic process. To render his representation of market participants' expectations consistent with the predictions of his own model, he represented their expectations of future outcomes with his model's conditional expectations of these outcomes. It was this implementation that came to be known as the rational expectations hypothesis (REH).

However, Muth's hypothesis neither presumes nor requires that outcomes are driven by a time-invariant stochastic process. What makes Muth's hypothesis central to macroeconomic and finance theory is that it applies to any economic model.

In contrast to REH models, our model acknowledges Knightian uncertainty on the part of the economist as the stochastic process assumed to drive dividends and earnings undergoes unforeseeable change. Muth's hypothesis implies that market participants understand that dividends and earnings evolve according to (1)-(3) and that they face Knightian uncertainty arising from unforeseeable change in μ_t.

As a consequence of unforeseeable change, our model's predictions of future dividends are given by the set of conditional expectations $\mathcal{E}\left(d_{t+i}|x_t; I^{\mu}\right)$, $i > 0$, in (8). As our model *ex ante* only constrains the future Knightian uncertainty parameters, $\mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}$, to lie within the interval I^{μ}, *any* of the conditional expectations in this set can serve as a model-consistent representation of market participants' expectations.

Building on the ambiguity-aversion literature, we assume in the next section that the consumption and portfolio choices of market participants facing Knightian uncertainty can be represented with a representative agent's maximization of intertemporal utility with multiple-priors. We represent this intertemporal utility over a set of expectations of future dividends and asset prices indexed by all combinations of the future Knightian uncertainty parameters within an interval. To implement Muth's hypothesis, we specify this interval to be a subset of the interval I^{μ}.

Our implementation of Muth's hypothesis formalizes the argument by Gajdos *et al.* (2005) and Epstein and Schneider (2010, p. 321) that ambiguity-averse individuals need not maximize their utility over the full set of conditional expectations that can characterize future outcomes. In our model, this set is constrained by the interval I^{μ}, which must necessarily be quite wide to encompass extreme unforeseeable changes that however rare, have led to abrupt shifts in dividends and earnings in the past and thus could do so again. This implies that to maximize intertemporal utility over the full set of conditional expectations, at all times t, an ambiguity-averse agent would behave extremely pessimistically, as if $\mu_{t+i} = \mu^L$ at all future times $i = 1, 2, \ldots$. Consequently, we represent the agent's expectations of future outcomes, at each time t, with a subset of the conditional expectations that constitute our model's predictions of these outcomes.

4.1 A Consistent Representation of Market Participants' Autonomous Expectations

Invoking Muth's hypothesis, the representative agent understands that the process driving dividends and earnings, as specified in (1)-(3), undergoes unforeseeable change, and that forming expectations about future dividends requires an assessment of the future Knightian uncertainty parameters. At each time t, we represent this assessment by an interval, J_t^μ, which we refer to as the agent's interval, and we represent the agent's expectations of d_{t+i}, $i > 0$, by the set of conditional expectations indexed by all combinations of $\mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}$ within J_t^μ.

Muth's hypothesis constrains the representation of an agent's assessment of this interval to be a subset of a model's interval, that is, $J_t^\mu \subseteq I^\mu$ at all times t. However, the key implication of unforeseeable change is that there is no fixed probabilistic or other objective way to assess the future Knightian uncertainty parameters. This implies that our model of how dividends and earnings unfold does not fully determine which subset of I^μ can represent the agent's assessment of the future Knightian uncertainty parameters at each point in time, given the available data, and how this subset is revised as new information becomes available. In this sense, the agent's interval, J_t^μ, is an autonomous input to our model.

Thus, opening the model to Knightian uncertainty enables us to represent the agent's expectations as being consistent with the predictions of our model and yet let them be autonomous.

This differentiates our application of Muth's hypothesis from its implementation in REH models. In these models, the time-invariant stochastic process driving outcomes fully determines the (single) model-consistent representation of market participants' expectations and how they are revised as new information becomes available over time.

To formally implement Muth's hypothesis, we specify the interval J_t^μ as

$$J_t^\mu = \left[\phi_t^L, \phi_t^U\right], \qquad \phi_t^L = \mu_t - \lambda_t^L, \qquad \phi_t^U = \mu_t + \lambda_t^U. \tag{9}$$

In general, the bounds of J_t^μ would be based on the agent's assessment of the value of μ_t. However, for the sake of simplicity, we assume that μ_t is known at time t and serves as an anchor for the interval's bounds, ϕ_t^L and ϕ_t^U.[6]

Additionally, we specify these bounds in terms of λ_t^L and λ_t^U, which represent the agent's assessment of future changes – particularly in the near future, which is not discounted as heavily as outcomes in the distant future – relative to μ_t. We assume that λ_t^L and λ_t^U are influenced by a variety of factors, including analyses of the available data and psychology. In the next subsection, we illustrate how the influence of market sentiment on the agent's 'expectations can be formalized through its influence on λ_t^L and λ_t^U.

Over time, the agent revises the interval J_t^μ in unforeseeable ways. The specification in (9) formalizes two sources of such revisions. First, unforeseeable changes in μ_t at times $\{\tau_j\}_{j=1}^{\infty}$ lead the agent to revise J_t^μ. Thus, the unforeseeable change that occurs in the process driving dividends and earnings feeds directly in to the agent's assessment of the interval within which future Knightian uncertainty parameters might lie.

Moreover, we assume that psychological influences, narratives, and market sentiment can lead the agent to revise J_t^μ even at times when μ_t does not change. Thus, we assume that λ_t^L and λ_t^U undergo unforeseeable change at times $\{\kappa_j\}_{j=1}^{\infty}$ with

$$\lambda_t^L = \bar{\lambda}_j^L, \qquad \lambda_t^U = \bar{\lambda}_j^U, \qquad \text{for } t = \kappa_{j-1}, \kappa_{j-1}+1, \ldots, \kappa_j - 1 \text{ and } j = 1, 2, \ldots, \tag{10}$$

where $\bar{\lambda}_j^L$ and $\bar{\lambda}_j^U$ are constant parameters, with either $\bar{\lambda}_j^L \neq \bar{\lambda}_{j-1}^L$ or $\bar{\lambda}_j^U \neq \bar{\lambda}_{j-1}^U$, $\kappa_0 = 1$, and $\kappa_{j-1} < \kappa_j$ for all $j = 1, 2, \ldots$. This specification implies that the bounds ϕ_t^L and ϕ_t^U in (9) undergo unforeseeable changes at times $\{\tau_j\}_{j=1}^{\infty}$ when μ_t changes and at times $\{\kappa_j\}_{j=1}^{\infty}$ when λ_t^L and λ_t^U change.

Although we specify the bounds of J_t^μ to depend on μ_t, which is part of the stochastic process driving dividends and earnings, the influence of λ_t^L and λ_t^U, together with the unforeseeable change in these parameters, implies that how the agent's interval is selected at a point in time and how it is revised over time are an autonomous input to the model.

To implement Muth's hypothesis, we restrict the parameters $\bar{\lambda}_j^L$ and $\bar{\lambda}_j^U$, such that

$$\bar{\lambda}_j^L \leq \bar{\mu}_i - \mu^L, \qquad \bar{\lambda}_j^U \leq \mu^U - \bar{\mu}_i, \qquad -\bar{\lambda}_j^L \leq \bar{\lambda}_j^U \qquad \text{for } j = 1, 2, \ldots. \tag{11}$$

Together with the restriction on $\bar{\mu}_i$ in (3), this constraint implies that $\mu^L \leq \phi_t^L \leq \phi_t^U \leq \mu^U$, such that $J_t^\mu \subseteq I^\mu$ at all times t. Thus, we constrain the agent's assessment of the interval for the future Knightian uncertainty parameters to be a subset of the interval within which they lie according to our model.

[6]In practice, μ_t would have to be estimated from the available data using some statistical method to estimate the breakpoints and parameters during different subperiods. Nevertheless, here we abstract from the considerable uncertainty regarding these estimates.

Given the interval J_t^μ specified in (9)-(11), we represent, at each time t, the agent's expectations of future dividends d_{t+i}, $i > 0$, by the set of conditional expectations indexed by all combinations of the future Knightian uncertainty parameters, $\mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}$, within J_t^μ. As the conditional expectation, $E\left(d_{t+i}|x_t; \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}\right)$, in (5) is monotonically increasing in the Knightian uncertainty parameters, this set of conditional expectations, which we denote by $\mathcal{E}\left(d_{t+i}|x_t; J_t^\mu\right)$, is given by

$$\begin{aligned}
\mathcal{E}\left(d_{t+i}|x_t; J_t^\mu\right) &= \left\{E\left(d_{t+i}|x_t; \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}\right) \mid \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i} \in J_t^\mu\right\} \\
&= \left[E\left(d_{t+i}|x_t; \phi_t^L\right), E\left(d_{t+i}|x_t; \phi_t^U\right)\right] \\
&= \beta\left(x_t\right)^{\rho^i} \omega_i \left[\exp\left(\sum_{j=1}^{i} \rho^{i-j}\left(\mu_t - \lambda_t^L\right)\right), \exp\left(\sum_{j=1}^{i} \rho^{i-j}\left(\mu_t + \lambda_t^U\right)\right)\right], \quad (12)
\end{aligned}$$

where ω_i is defined in (6). It follows from $J_t^\mu \subseteq I^\mu$ that $\mathcal{E}\left(d_{t+i}|x_t; J_t^\mu\right) \subseteq \mathcal{E}\left(d_{t+i}|x_t; I^\mu\right)$, so we represent the agent's expectations of future dividends with a subset of our model's predictions of these.

Our representation of the agent's set of expectations of future dividends in (12) is based on current earnings, x_t, while the bounds of the set depend on $\phi_t^L = \mu_t - \lambda_t^L$ and $\phi_t^U = \mu_t + \lambda_t^U$. Over time, the set of expectations undergo unforeseeable change arising from unforeseeable change in earnings. Moreover, because ϕ_t^L and ϕ_t^U undergo unforeseeable change, the mapping of current earnings onto a set of expectations of future dividends also undergoes unforeseeable change.

4.2 Reconciling the Influence of Market Sentiment with Muth's Hypothesis

Our model-consistent representation under Knightian uncertainty shares a key feature with REH models: the agent's expectations are driven by fundamentals, specifically current earnings x_t. However, in contrast to REH models, which represent the agent's expectations as driven *solely* by fundamental factors, allowing for an autonomous role of expectations enables us to formalize the influence of *both* fundamental and non-fundamental factors, while maintaining Muth's hypothesis.

Our representation in (12) reduces to REH if the unforeseeable change in μ_t is replaced by the assumption that this parameter is constant over time or changes according to a probabilistic rule. In the former case, the assumption $\mu_t = \mu$ at all t implies that the stochastic process in (1)-(3) becomes time-invariant, and thus that the Knightian uncertainty about future dividends reduces to standard probabilistic risk. As a consequence, the interval

J_t^{μ} in (9)-(11) reduces to $J_t^{\mu} = \mu$ for all t, and therefore the representation in (12) reduces to REH:

$$E\left(d_{t+i}|x_t;\mu\right) = \beta\left(x_t\right)^{\rho^i}\exp\left(\sum_{j=1}^{i}\rho^{i-j}\mu\right)\omega_i. \tag{13}$$

This representation of the agent's expectations is fully determined by the time-invariant stochastic process driving dividends and earnings. As (13) makes clear, REH represents the agent's expectations as driven solely by earnings, x_t, thereby excluding the influence of other factors.

As in REH models, the implementation of Muth's hypothesis in a model with resolvable ambiguity would lead to a representation of the agent's expectations that is solely driven by fundamentals. Our model can, for example, be reduced to a simple model with resolvable ambiguity by replacing the unforeseeable change in μ_t in (2)-(3) with the assumption that $\mu_t = \mu$ at all times t, where $\mu \in I^{\mu} = \left[\mu^L, \mu^U\right]$. In this case, the ambiguity about μ would be resolvable in the sense that its value could be consistently estimated from the data, such that the ambiguity vanished asymptotically.

For example, Epstein and Schneider (2007) show that a model for an agent with multiple-priors utility, for example with an initial prior for μ over the interval I^{μ}, could indeed resolve such ambiguity asymptotically. In this case, implementing Muth's hypothesis would mean that the agent's expectations must be based on a mechanisms that resolves the ambiguity asymptotically. This would imply that the model-consistent representation of the agent's expectations would be driven solely by earnings over time.

In contrast to REH models, as well as those with resolvable ambiguity, behavioral-finance models have focused on the role of psychological and other non-fundamental factors in driving market participants' expectations and thereby aggregate outcomes, such as asset prices.

Importantly, behavioral-finance models have formalized the influence of such factors on the agent's expectations and the resulting aggregate outcomes with time-invariant stochastic processes. Because such a representation of the agent's expectations is inconsistent with the predictions of the economist's own model for how outcomes unfold, it violates Muth's hypothesis. Consequently, in models representing outcomes with a time-invariant stochastic process, the influence of psychological factors has been interpreted as a symptom of participants' irrationality.

Under Knightian uncertainty, REH does not provide the standard of rationality. Indeed, Muth's hypothesis implies that an agent would not base his expectations on a time-invariant stochastic process.

However, a utility-maximizing agent facing Knightian uncertainty must form expectations about future outcomes upon which to base his decisions. According to our model, this requires selecting a specific set of scenarios for the future Knightian uncertainty parameters, as represented by the agent's interval, J_t^{μ}, within the many scenarios that are possible *ex ante*. Herein lies the key importance of leaving the model open to Knightian uncertainty: psychological and other non-fundamental factors, such as market sentiment and narrative market accounts, influence which of the many possible model-consistent expectations represent the agent's expectations.[7]

We illustrate this idea by assuming that the agent's selection of the interval, J_t^{μ}, is influenced by market sentiment. Specifically, let s_t denote a market-sentiment index, such that the market is "optimistic" when $s_t = 1$, "neutral" when $s_t = 0$, and "pessimistic" when $s_t = -1$. We assume that an optimistic (pessimistic) market sentiment leads the agent to shift upward (downward) the interval J_t^{μ}, as represented by λ_t^L and λ_t^U in (9)-(11). However, in contrast to the probabilistic formalization of psychological factors in behavioral models, we assume that market sentiment and its influence on the agent's expectations change in unforeseeable ways.

We formalize this influence qualitatively by assuming that the direction of the change in $\bar{\lambda}_j^L$ and $\bar{\lambda}_j^U$ at times $\{\kappa_j\}_{j=1}^{\infty}$ depends on whether market sentiment is optimistic or pessimistic, as given by

$$\begin{array}{lllll} \bar{\lambda}_j^L > \bar{\lambda}_{j-1}^L & \text{and} & \bar{\lambda}_j^U > \bar{\lambda}_{j-1}^U & \text{if } s_t = 1, \\ \bar{\lambda}_j^L < \bar{\lambda}_{j-1}^L & \text{and} & \bar{\lambda}_j^U < \bar{\lambda}_{j-1}^U & \text{if } s_t = -1, \end{array} \tag{14}$$

where $\bar{\lambda}_j^L$ and $\bar{\lambda}_j^U$ are specified in (10).

Thus, optimistic market sentiment at time $t = \kappa_j$, $s_{\kappa_j} = 1$, leads the agent to revise upward his assessment of the future Knightian uncertainty parameters. This shifts the bounds of the interval J_t^{μ} upward, which, *ceteris paribus*, implies that the agent revises upward his set of expectations for future dividends. Depending on whether earnings increase or decrease at time $t = \kappa_j$, the influence of market sentiment might reinforce or even offset the direct effect from earnings on the agent's expectations. For example, this specification allows for a scenario in which pessimistic market sentiment leads to a downward revision of the agent's expectations despite increasing earnings.

[7] Akerlof and Snower (2016) and Shiller (2017, 2019) provide extensive discussions of the importance of narrative accounts in understanding market outcomes. Mangee (2021) provides extensive empirical evidence of the role of narratives in driving market participants' expectations and aggregate outcomes under Knightian uncertainty arising from unforeseeable change.

5 Intertemporal Utility Under Knightian Uncertainty

We extend Lucas's (1978) general equilibrium asset-price model by introducing both Knightian uncertainty arising from unforeseeable change and ambiguity aversion. To this end, we assume that market participants' portfolio and consumption choices can be represented with the outcomes of a representative agent's intertemporal utility maximization. As the agent faces Knightian uncertainty, we assume that he has intertemporal utility with multiple-priors and we specify this utility over a set of model-consistent expectations of future dividends and asset prices. We allow for unforeseeable change in the agent's expectations over time, implying that the agent's optimal consumption and portfolio choices also change in unforeseeable ways.

Specifically, at each time t, we consider the intertemporal utility function defined over the horizon $t+i$ for $i = 1, 2, \ldots$. This intertemporal utility is specified over the set of conditional expectations indexed by all combinations of the future Knightian uncertainty parameters, $\mu_{t+1}, \mu_{t+2}, \ldots \in J_t^\mu$, with J_t^μ specified in (9)-(11) and (14), as given by

$$\min_{\mu_{t+1},\mu_{t+2},\ldots \in J_t^\mu} \left(\sum_{i=0}^{\infty} \gamma^i E\left(u\left(c_{t+i}\right) | x_t; \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}\right) \right), \tag{15}$$

where $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, bounded, increasing, strictly concave, and with $u(0) = 0$, $0 < \gamma < 1$ and $E\left(u\left(c_{t+i}\right) | x_t; \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}\right)$ denoting the expected utility of consumption c_{t+i}, $i \geq 0$, given x_t and indexed by $\mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}$. Although the representative agent considers the set of expected utilities indexed by all combinations of $\mu_{t+1}, \mu_{t+2}, \ldots \in J_t^\mu$, his ambiguity aversion implies that his utility depends only on the minimum conditional expectation over this set.

We assume that the agent has an exogenous endowment, e_t, and can buy a single asset at the price, p_t, where the asset-price function is exogenous. The asset pays dividends, d_t, at the beginning of period t. Thus, at each t, the agent must decide consumption, $c_{t+i} \geq 0$, and the end-of-period amount of assets to buy, δ_{t+i}, $i = 1, 2, \ldots$, as a function of the realized future values, $\{e_{t+j}, d_{t+j}, p_{t+j}\}_{j=1}^{i}$, subject to the budget constraints,

$$c_{t+i} + \delta_{t+i} p_{t+i} \leq e_{t+i} + \delta_{t+i-1}\left(d_{t+i} + p_{t+i}\right), \qquad \text{for all } i \geq 0,\ 0 \leq \delta_{t+i} \leq \bar{\delta}, \tag{16}$$

and given the value of δ_{t-1}, where $\bar{\delta} \geq 1$ is an upper bound on the asset holding. As higher future dividends and asset-prices makes higher consumption possible in the future, the intertemporal utility depends on expected future dividends and asset prices.

Using the properties of u and the monotonicity of the conditional expectation of future dividends with respect to $\mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}$, the intertemporal utility function in (15) can be rewritten as

$$\begin{aligned}
& \min_{\mu_{t+1}, \mu_{t+2}, \ldots \in J_t^{\mu}} \left(\sum_{i=0}^{\infty} \gamma^i E\left(u\left(c_{t+i}\right) | x_t; \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}\right) \right) \\
= \; & \min \sum_{i=0}^{\infty} \gamma^i \left\{ E\left(u\left(c_{t+i}\right) | x_t; \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i}\right) \mid \mu_{t+1}, \mu_{t+2}, \ldots, \mu_{t+i} \in J_t^{\mu} \right\} \\
= \; & \sum_{i=0}^{\infty} \gamma^i E\left(u\left(c_{t+i}\right) | x_t; \phi_t^L\right), \qquad (17)
\end{aligned}$$

where $E\left(u\left(c_{t+i}\right) | x_t; \phi_t^L\right)$ and $E\left(u\left(c_{t+i}\right) | x_t; \phi_t^L\right)$ denote the expected utilities indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \mu_{t+i} = \phi_t^L$ and $\mu_{t+1} = \mu_{t+2} = \ldots = \mu_{t+i} = \phi_t^U$, respectively. We summarize this result with the following lemma.

Lemma 1 *Given the stochastic process for dividends and earnings in (1)-(3) and the interval J_t^{μ} in (9)-(11) and (14), the intertemporal utility function in (15) can be rewritten as*

$$\sum_{i=0}^{\infty} \gamma^i E\left(u\left(c_{t+i}\right) | x_t; \phi_t^L\right), \qquad (18)$$

where $E\left(u\left(c_{t+i}\right) | x_t; \phi_t^L\right)$ denotes the conditional expected utility indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \mu_{t+i} = \phi_t^L$ for $i > 0$.

Lemma 1 shows that the multiple-priors intertemporal utility function in (15) can be rewritten as the standard intertemporal utility function in (18) indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L$. That is, at time t, the ambiguity-averse agent acts as if he maximizes a standard intertemporal utility function based on the worst-case expectation that future dividends and earnings $\{d_{t+i}, x_{t+i}\}_{i=1}^{\infty}$ unfold according to

$$d_{t+i} = \beta x_{t+i}, \qquad \log x_{t+i} = \rho \log x_{t+i-1} + \phi_t^L + \varepsilon_{t+i}, \qquad (19)$$

for $i = 1, 2, \ldots$.

As viewed from time t, this is a stationary process for future dividends and earnings $\{d_{t+i}, x_{t+i}\}_{i=1}^{\infty}$, given (d_t, x_t). Thus, while the actual process for $\{d_t, x_t\}_{t=1}^{\infty}$ in (1)-(3) undergoes unforeseeable change in μ_t, the intertemporal utility function in (18) depends, at each time t, on the worst-case expectations corresponding to the conditional expectations of the stationary process for future dividends and earnings $\{d_{t+i}, x_{t+i}\}_{i=1}^{\infty}$ in (19).

Building on Lucas (1978), this enables us to define the optimal value function corresponding to the intertemporal utility function in (18). This optimal value function, expressed as a function of current earnings, x_t, and the past portfolio, δ_{t-1}, and based on worst-case expectations indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L$, is given by

$$v\left(\delta_{t-1}, x_t; \phi_t^L\right) = \max_{c_t, \delta_t} \left(u\left(c_t\right) + \gamma E\left(v\left(\delta_t, x_{t+1}; \phi_t^L\right)\right)\right), \tag{20}$$

subject to the budget constraint in (16). Given last period's portfolio, δ_{t-1}, current earnings, x_t, and worst-case expectations indexed by $\phi_t^L = \mu_t - \lambda_t^L$, v in (20) is the utility obtained from the optimal consumption and portfolio plan at time t, assuming that the optimal plan chosen at time $t+1$ is based on worst-case expectations that are also indexed by $\mu_{t+2} = \mu_{t+3} = \ldots = \phi_t^L$.

When the agent revises his interval for the future Knightian uncertainty parameters from time t to $t+1$, i.e. $\phi_t^L \neq \phi_{t+1}^L$, such that $J_t^\mu \neq J_{t+1}^\mu$, he revises his expectations in an unforeseeable way, which in turn leads him to solve a new utility maximization problem at time $t+1$. That leads him to revise his optimal consumption and portfolio plan in an unforeseeable way, relative to his optimal plan at time t.

To illustrate this point formally, consider the unforeseeable change in $\phi_t^L = \mu_t - \lambda_t^L$, as specified in (10), from time $t = \kappa_j - 1$ to $t+1 = \kappa_j$, for some $j > 0$. Here, ϕ_t^L changes from $\phi_t^L = \bar{\mu}_i - \bar{\lambda}_j^L$ to $\phi_{t+1}^L = \bar{\mu}_i - \bar{\lambda}_{j+1}^L$. Thus, as specified by the optimal value function in (20), the agent chooses his optimal consumption and portfolio at time t, (c_t, δ_t), based on worst-case expectations indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L = \bar{\mu}_i - \bar{\lambda}_j^L$. However, the change in ϕ_t^L from time t to $t+1$ leads him to revise his optimal consumption-portfolio plan, relative to the optimal plan made at time t, as it is now based on worst-case expectations indexed by $\mu_{t+2} = \mu_{t+3} = \ldots = \phi_{t+1}^L = \bar{\mu}_i - \bar{\lambda}_{j+1}^L$.

6 Equilibrium Asset Prices

As in Lucas (1978), we define the general equilibrium at each point in time in terms of an optimal value function and an asset-price function. Extending Lucas's model to allow for both Knightian uncertainty arising from unforeseeable change and ambiguity aversion, both of these functions are based on the representation of the agent's worst-case expectations of future dividends and asset prices. These are indexed by the future Knightian uncertainty parameters taking the value $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L$, where ϕ_t^L represents the lower bound of the agent's interval for these parameters, as viewed from time t. This assessment implies that the worst-case expectations are based on $\{d_{t+i}\}_{i=1}^{\infty}$ evolving according to the stationary stochastic process in (19). Given this representation of worst-case expectations, we formally define the general equilibrium as follows:

Definition 1 *Given the stochastic process for future dividends,* $\{d_{t+i}\}_{i=1}^{\infty}$ *in (19), implied by the representative agent's assessment of the interval for the future Knightian uncertainty parameters with lower bound* ϕ_t^L *in (9), we define the general equilibrium at each point in time in terms of the pair of functions*

$$v\left(\delta_{t-1}, x_t; \phi_t^L\right), \qquad \textit{and} \qquad p\left(x_t; \phi_t^L\right), \tag{21}$$

where $v\left(\delta_{t-1}, x_t; \phi_t^L\right)$ *is specified in (20), and subject to the budget constraint in (16). The general equilibrium at time* t *is attained by the market-clearing condition,* $\delta_t = 1$.

Because expectations, once indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L$, are based on the stationary process for $\{d_{t+i}\}_{i=1}^{\infty}$, we can directly apply Lucas's (1978) approach to characterizing the asset-price function in general equilibrium. We state the resulting stochastic Euler equation with the following theorem:

Theorem 1 *The general equilibrium in Definition 1 implies that the asset-price function* $p\left(x_t; \phi_t^L\right)$ *satisfies the stochastic Euler equation indexed by* $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L$*, given by*

$$p_t = p\left(x_t; \phi_t^L\right) = \gamma E\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left(d_{t+1} + p\left(x_{t+1}; \phi_t^L\right)\right)|x_t; \phi_t^L\right), \tag{22}$$

at each time $t = 1, 2, \ldots$.

Assuming a specific functional form for the instantaneous utility function, u, the stochastic Euler equation in (22) can be solved for an explicit expression of the asset-price function $p\left(x_t; \phi_t^L\right)$. In the next subsection, we illustrate this for the simple case where u is linear.

Given the worst-case expectations indexed by $\mu_{t+1} = \mu_{t+2} = \ldots = \phi_t^L$, our model implies that the asset-price function is determined by (22) at each point in time. In contrast, Epstein and Wang's (1994) extension of Lucas's (1978) intertemporal model with ambiguity implies asset-price indeterminacy, in the sense that prices can be characterized by an interval of stochastic Euler equations. Epstein and Wang assume, as we do, that the representative agent has intertemporal utility with multiple-priors specified over a set of distributions for future dividends. The asset-price indeterminacy in their model arises in situations where the minimum expected utility is not uniquely determined by one of the distributions in this set.

As Lemma 1 implies, this minimum is uniquely determined in our model, because the conditional expectation of future dividends is monotonically increasing with respect to the Knightian uncertainty parameters. Consequently, the asset-price is uniquely determined by the stochastic Euler equation in (22) at each point in time.

The novel implication of our model is that the stochastic Euler equation in (22) undergoes unforeseeable change over time, owing to unforeseeable change in earnings, x_t, at times $\{\tau_j\}_{j=1}^{\infty}$ where μ_t changes, as well as to unforeseeable change in the agent's worst-case expectations of future dividends and asset prices. At time t, these are based on the agent's interval for the future Knightian uncertainty parameters with lower bound $\phi_t^L = \mu_t - \lambda_t^L$. Because the agent revises this interval in unforeseeable ways at times $\{\tau_i, \kappa_j\}_{i,j=1}^{\infty}$, he revises his worst-case expectations. As a result, the optimal value function is revised as well, which leads to unforeseeable changes in the stochastic Euler equation characterizing the asset price. These changes are unforeseeable in the sense that the agent could not take them into account when forming expectations and making optimal decisions in the past.

As a consequence of these unforeseeable changes, our model implies Knightian uncertainty about future asset prices: the stochastic Euler equation that characterizes $p\left(x_{t+i}; \phi_{t+i}^L\right)$, $i > 0$, is unknown at all t. However, invoking Muth's hypothesis, the asset-price function implied by (22) is bounded, because the specification of ϕ_t^L implies that $\mu^L \leq \phi_t^L \leq \mu^U$, where μ^L and μ^U are the bounds of the interval within which the Knightian uncertainty parameter μ_t lies, according to our model. We state this result with the following corollary.

Corollary 1 *The general equilibrium asset-price function $p\left(x_t; \phi_t^L\right)$ in (22) is bounded by the interval of stochastic Euler equations given by*

$$
\begin{aligned}
p\left(x_t; \phi_t^L\right) \quad \in \quad & \left[\gamma E\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left(d_{t+1} + p\left(x_{t+1}; \mu^L\right)\right) | x_t; \mu^L\right),\right. \\
& \left.\gamma E\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left(d_{t+1} + p\left(x_{t+1}; \mu^U\right)\right) | x_t; \mu^U\right)\right], \qquad (23)
\end{aligned}
$$

at all times $t = 1, 2, \ldots$.

It follows from this corollary that the asset price can be characterized *ex ante* by any of the price functions in the interval in (23). Which of them actually characterizes the price at time t is determined by ϕ_t^L. Moreover, because ϕ_t^L changes in unforeseeable ways, our model formalizes the economist's unresolvable ambiguity about future asset prices.

6.1 Example: Linear Utility

We consider an example with a linear instantaneous utility function, $u\left(c_t\right) = c_t$. Although it violates the boundary condition for u used above, as noted by Lucas (1978, p. 1439), this special case can be handled separately and implies that the stochastic Euler equation in (22) reduces to

$$
p\left(x_t; \phi_t^L\right) = \gamma E\left(d_{t+1} | x_t; \phi_t^L\right) + \gamma E\left(p\left(x_{t+1}; \phi_t^L\right) | x_t; \phi_t^L\right). \qquad (24)
$$

As the conditional expectations on the right-hand side are indexed by ϕ_t^L, we can iterate the expression forward to derive a closed-form solution for $p_t = p\left(x_t; \phi_t^L\right)$. To this end, we insert

$$p\left(x_{t+1}; \phi_t^L\right) = \gamma E\left(d_{t+2} | x_{t+1}; \phi_t^L\right) + \gamma E\left(p\left(x_{t+2}; \phi_t^L\right) | x_{t+1}; \phi_t^L\right),$$

into (24). Continuing such iterations, and assuming a standard transversality condition, we obtain the present-value expression for the asset price at time t, given ϕ_t^L:

$$p_t = p\left(x_t; \phi_t^L\right) = \sum_{i=1}^{\infty} \gamma^i E\left(d_{t+i} | x_t; \phi_t^L\right) = \sum_{i=1}^{\infty} \gamma^i \beta \exp\left(\sum_{j=1}^{i} \rho^{i-j} \phi_t^L\right) \omega_i\left(x_t\right)^{\rho^i}. \qquad (25)$$

This explicit expression shows that the asset price depends on current earnings and $\phi_t^L = \mu_t - \lambda_t^L$, which denotes the lower bound of the agent's assessment of the interval for the future Knightian uncertainty parameters.

Over time, the asset-price function in (25) undergoes unforeseeable change: the timing and magnitude of this change is unknown *ex ante*. This arises from unforeseeable change in earnings x_t, in μ_t at times $\{\tau_j\}_{j=1}^{\infty}$, as specified in (2)-(3), and in the agent's worst-case expectations through unforeseeable change in $\phi_t^L = \mu_t - \lambda_t^L$ at times $\{\tau_i, \kappa_j\}_{i,j=1}^{\infty}$, as specified in (9)-(11), (14), and (2).

7 The Autonomous Role of Expectations in Driving Asset Prices

Opening the model to Knightian uncertainty enables us to accord the agent's expectations an autonomous role in driving asset prices, while adhering to Muth's hypothesis. Over time, because our model-consistent representation of the agent's worst-case expectations of future dividends and asset prices is based on earnings, the asset price in (25) is also driven by earnings. As earnings undergo unforeseeable change arising from μ_t, the asset price undergoes unforeseeable change.

Importantly, the asset price is also driven by changes in ϕ_t^L. At each point in time, ϕ_t^L formally represents how the agent maps current earnings onto worst-case expectations of future dividends and asset prices. As ϕ_t^L undergoes unforeseeable change, this mapping changes in unforeseeable ways. In this sense, the agent's expectations play an autonomous role in driving the asset price over time.

Opening the model to Knightian uncertainty arising from unforeseeable change thus enables us to formalize the key insight of Phelps's (1970) micro-foundations approach: market participants' expectations play an autonomous role in driving aggregate outcomes. This leads to our model's novel prediction that the *relationship* between the asset price and earnings undergoes unforeseeable change, which occurs at times $\{\tau_i, \kappa_j\}_{i,j=1}^{\infty}$, when the unforeseeable change in ϕ_t^L changes the relationship between the asset price in (25) and earnings.

This prediction substantially differs from REH models, such as Lucas's. Imposing consistency within models that represent outcomes with a time-invariant stochastic process, as REH models do, implies that expectations do not play an autonomous role in driving asset prices: they are fully determined by the time-invariant dividend process. As a result, the relationship between the model's input and output variables – earnings and the asset price – remains constant over time.

To illustrate this, recall from above that our model reduces to an REH model if the unforeseeable change in μ_t is replaced by the assumption that $\mu_t = \mu$ at all times. That would imply that $\phi_t^L = \phi_t^U = \mu$, such that the relationship between the asset price and earnings in (25) becomes constant over time.

In Ilut and Schneider's (2014) New Keynesian business model with ambiguity arising from unforeseeable change, the ambiguity-averse agent's worst-case expectations are represented as autonomous, as in our model. They represent the analog to the lower bound of the agent's interval for the future Knightian uncertainty parameters, ϕ_t^L in our model, as evolving over time according to a stationary autoregressive process.

This implies that the model's output variable (the asset price in our model) undergoes unforeseeable change arising from the unforeseeable change in the input variable (earnings). However, as in REH models, and in contrast to our approach, the relationship between the two variables would *not* undergo unforeseeable change, because it evolves according to a stationary autoregressive process around the time-invariant mean.

7.1 The Influence of Both Fundamentals and Market Sentiment in Driving Asset-prices

As in REH models, our model-consistent representation of the representative agent's expectations is based on fundamentals, specifically earnings. However, while agents' expectations are driven solely by fundamental factors in REH models, representing them as autonomous allows us to formalize the influence of both fundamental and non-fundamental factors on expectations and thus on how the asset price unfolds over time. In contrast to behavioral-finance models, we do so while maintaining Muth's hypothesis.

A central implication of our model is that psychological factors, such as market sentiment, can either reinforce, dampen, or even outweigh the effect of a change in earnings on the asset price. Consider, for example, a situation where a positive random shock ε_t leads to an increase in dividends and earnings at time $t = \kappa_j$ for some $j > 0$, while $\mu_t = \mu_{t-1} = \bar{\mu}_i$ for some $i > 0$. *Ceteris paribus*, this increases the asset price in (25). However, pessimistic market sentiment dampens or even outweighs this effect, because such sentiment at time $t = \kappa_j$ implies that $\bar{\lambda}_j^L < \bar{\lambda}_{j-1}^L$ according to (14), which, *ceteris paribus*, leads to a decrease in the asset price as $\phi_t^L = \bar{\mu}_i - \bar{\lambda}_j^L < \phi_{t-1}^L = \bar{\mu}_i - \bar{\lambda}_{j-1}^L$. As our model specifies the effect of market sentiment on the agent's expectations only in qualitative terms, it does not specify *ex ante* which of these two effects will dominate.

Analogously, the positive effect of higher dividends and earnings on the asset price can be reinforced by unforeseeable change in the agent's expectations if market sentiment is optimistic at time $t = \kappa_j$ for some $j > 0$.

These implications are consistent with the empirical findings by Frydman, Mangee, and Stillwagon (2021). They show that market sentiment influences market participants' forecasts of stock returns: their optimism (pessimism) affects the weights they assign to fundamentals. Importantly, they also find that the influence of market sentiment on participants' forecasts of stock returns is highly irregular, both in timing and magnitude.

For example, if good (bad) "news" about dividends and interest rates coincides with market optimism (pessimism), the news about these fundamentals has a significant effect on participants' forecasts of future returns and has the expected signs. These findings support our hypothesis that the influence of market sentiment changes in unforeseeable ways.

8 Concluding Remarks

Knight (1921, pp. 198, 231-232) argued that change in the economy's structure is caused, at least intermittently, by non-repetitive events, and thus cannot be foreseen *ex ante* with a probabilistic rule, such as Markov switching. As a result of such unforeseeable change, any time-invariant stochastic process that represents outcomes eventually becomes inconsistent with time-series data.

For Knight, recognizing unforeseeable change is the key to understanding profit-seeking activities and the resulting market outcomes. Here, we have proposed a tractable approach to building intertemporal macroeconomic and finance models that is premised on Knight's insight. We do so by opening Lucas's (1978) seminal model to unforeseeable change in the stochastic process for dividends and asset prices. This formalizes one of the two pillars of our approach: that a model-builder faces Knightian uncertainty about future outcomes.

The second pillar of our approach to modeling aggregate outcomes under Knightian uncertainty is the premise that market participants are rational: they are profit-seeking and relate their expectations to some understanding, however imperfect, of the process driving outcomes. In order to formalize optimal decisions under Knightian uncertainty, we extend the ambiguity-aversion approach to situations in which the process driving outcomes undergoes unforeseeable change. We represent participants' understanding of this process by implementing Muth's (1961) pathbreaking model-consistency hypothesis when both a model-builder and market participants face Knightian uncertainty.

As in REH models, imposing consistency within a model open to unforeseeable change relates participants' expectations of aggregate outcomes to fundamentals. However, in contrast to REH models, our approach implies a novel prediction: the relationship between prices and dividends undergoes unforeseeable change.

Remarkably, introducing Knightian uncertainty into the Lucas model reconciles Muth's hypothesis, which underpins REH, with behavioral-finance models' premise that non-fundamental factors, such as market sentiment, exert an autonomous influence on participants' expectations. In contrast to behavioral-finance models, however, our approach formalizes the influence of such factors without presuming that market participants are irrational.

Recognizing uncertainty that cannot be represented with standard probabilistic measures of "risk" is increasingly viewed as crucial to remedying the shortcomings of macroeconomic and finance theory. For example, in his Nobel lecture, Hansen (2013, p. 399) argues that REH models "miss something *essential*: uncertainty [arising from] ambiguity about which is the correct model" of the process driving aggregate outcomes (emphasis added).

Models assuming ambiguity typically formalize it with a set of time-invariant stochastic processes. Because such formalizations ignore unforeseeable change in the process driving outcomes, we argue that that they are incompatible with Muth's hypothesis. In contrast, our approach reconciles Muth's hypothesis with the premise that market participants face ambiguity about the process driving outcomes.

As our extension of the Lucas model shows, recognizing that an economist and market participants face Knightian uncertainty provides a way to build macroeconomic and finance models that are based on the key insights underpinning the milestone approaches developed since the 1970s. All of them – Phelps's (1970) micro-foundations approach, REH, behavioral-finance models, and the ambiguity-aversion approach – are mutually incompatible, on logical grounds, when the stochastic process driving outcomes is assumed to be time-invariant.

Opening models to Knightian uncertainty and combining these approaches' essential insights is crucial, we believe, to enhancing our understanding of how macroeconomic and financial outcomes driven by market participants' expectations unfold over time. Although we implement our approach in the context of the Lucas model of asset prices, economists and market participants face Knightian uncertainty about other aggregate outcomes, such as productivity, inflation, and unemployment. The development of models that recognize this is a task of future research.

References

[1] Akerlof, George A., and Dennis J. Snower. 2016. "Bread and bullets," *Journal of Economic Behavior & Organization*, 126, p. 58–71.

[2] Ang, Andrew, and Allan Timmermann. 2012. "Regime changes and financial markets," *Annual Review of Financial Economics,* 4, p. 313–327.

[3] Bai, Jushan, and Pierre Perron. 1998. "Estimating and testing linear models with multiple structural changes," *Econometrica*, p. 47–78.

[4] Barberis, Nicholas, Andrei Shleifer, and Robert Vishny. 1998. "A model of investor sentiment," *Journal of Financial Economics*, 49, p. 307–343.

[5] Castle, Jennifer L., Jurgen A. Doornik, and David F. Hendry. 2012. "Model selection when there are multiple breaks," *Journal of Econometrics*, p. 239–246.

[6] Castle, Jennifer L., Jurgen A. Doornik, David F. Hendry, and Felix Pretis. 2015. "Detecting location shifts during model selection by step-indicator saturation," *Econometrics* 3 (2), p. 240–264.

[7] Cecchetti, Stephen G., Pok-Sang Lam, and Nelson C. Mark. 1990. "Mean reversion in equilibrium asset prices," *The American Economic Review*, 80 (3), p. 398–418.

[8] Ellsberg, Daniel. 1961. "Risk, ambiguity, and the Savage axioms," *The Quarterly Journal of Economics*, 75, p. 643–669.

[9] Epstein, Larry G., and Martin Schneider. 2003. "Recursive multiple-priors," *Journal of Economic Theory*, 113, p. 1–31.

[10] Epstein, Larry G., and Martin Schneider. 2007. "Learning under ambiguity," *Review of Economic Studies*, 74, p. 1275–1303.

[11] Epstein, Larry G., and Martin Schneider. 2010. "Ambiguity and asset markets," *Annual Review of Financial Economics*, 2, p. 315–346.

[12] Epstein, Larry G., and Tan Wang. 1994. "Intertemporal asset pricing under Knightian uncertainty," *Econometrica*, 62 (2), p. 283–322.

[13] Frydman, Roman, and Michael D. Goldberg. 2007. *Imperfect Knowledge Economics.* Princeton University Press.

[14] Frydman, Roman and Michael D. Goldberg. 2013. "The imperfect knowledge imperative in modern macroeconomics and finance theory," in Roman Frydman and Edmund S. Phelps (eds.), *Rethinking expectations: The way forward for macroeconomics*, Princeton University Press, p. 130–165.

[15] Frydman, Roman, and Edmund S. Phelps. 2013. "Which way forward for macroeconomics and policy analysis?" in Frydman, Roman, and Edmund S. Phelps (eds.), *Rethinking expectations: The way forward for macroeconomics*, Princeton University Press, p. 1–48.

[16] Frydman, Roman, Nicholas Mangee, and Josh Stillwagon. 2020. "How market sentiment drives forecasts of stock returns," *Journal of Behavioral Finance*, p. 1–17.

[17] Gajdos, Thibaut., Takashi Hayashi, Jean-Marc Tallon, and Jean Christophe Vergnaud. 2008. "Attitude towards imprecise information," *Journal of Economic Theory*, 140, p. 27–65.

[18] Gennaioli, Nicola, and Andrei Shleifer. 2018. *A Crisis of Beliefs: Investor Psychology and Financial Fragility*. Princeton University Press.

[19] Gilboa, Itzhak, and David Schmeidler. 1989. "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18, p. 141–153.

[20] Guidolin, Massimo, and Francesca Rinaldi. 2013. "Ambiguity in asset pricing and portfolio choice: a review of the literature," *Theory and Decision*, 74, p. 183–217.

[21] Hamilton, James D. 1988. "Rational-expectations econometric analysis of changes in regime: An investigation of the term structure of interest rates," *Journal of Economics Dynamics and Control*, 12, p. 385–423.

[22] Hamilton, James D. 1989. "A new approach to the economic analysis of nonstationary time series and the business cycle," *Econometrica*, 57, p. 357–84.

[23] Hansen, Lars Peter. 2013. "Uncertainty outside and inside economic models," The Nobel Prize Lecture, The Nobel Foundation.

[24] Ilut, Cosmin L., and Martin Schneider. 2014. "Ambiguous business cycles," *American Economic Review*, 104, p. 2368–2399.

[25] Knight, Frank H. 1921. *Risk, Uncertainty and Profit*. Boston: Houghton Mifflin.

[26] Lucas, Robert E. Jr. 1978. "Asset prices in an exchange economy," *Econometrica*, 46, p. 1429–1445.

[27] Mangee, Nicholas. 2021. *How Novelty and Narratives Drive the Stock Market: Black Swans, Animal Spirits and Scapegoats*. Cambridge University Press.

[28] Muth, John F. 1961. "Rational expectations and the theory of price movements," *Econometrica*, 29, 315-335.

[29] Pettenuzzo, Davide, and Allan Timmermann. 2011. "Predictability of stock returns and asset allocation under structural breaks," *Journal of Econometrics*, 164, p. 60–78.

[30] Phelps, Edmund S. 1970. "Introduction," in Edmund S. Phelps *et al.* (eds.), *Microeconomic Foundations of Employment and Inflation Theory*, Norton.

[31] Shiller, Robert J. 2017. "Narrative economics," *American Economic Review*, 107 (4), p. 967–1004.

[32] Shiller, Robert J. 2019. *Narrative Economics: How Stories Go Viral and Drive Major Economic Events*. Princeton University Press.

[33] Shleifer, Andrei. 2000. *Inefficient Markets: An Introduction to Behavioural Finance.* Oxford University Press.

[34] Timmermann, Allan. 2001. "Structural breaks, incomplete information, and stock prices," *Journal of Business & Economic Statistics*, 19 (3), p. 299–314.