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# Muth's Hypothesis Under Knightian Uncertainty: A Novel Account of Inflation Forecasts*

## Roman Frydman,† and Morten Nyboe Tabor‡

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**Abstract**

We open a New Keynesian Phillips curve model to nonrecurring structural shifts in its parameters and propose a novel implementation of Muth's hypothesis to represent market participants' inflation expectations under Knightian uncertainty arising from such shifts. We refer to our approach as the Knight-Muth hypothesis (KMH). We find empirical support for KMH's core premise that processes driving inflation time-series and inflation forecasts undergo nonrecurring structural shifts. In contrast to the rational expectations hypothesis and behavioral specifications, KMH reconciles model consistency with an autonomous role for participants' expectations in driving aggregate outcomes and the influence of psychological factors on those expectations.

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† Department of Economics, New York University. E-mail: rf3@nyu.edu.
‡ Senior Research Associate, Institute for New Economic Thinking (INET). E-mail: mortentabor@gmail.com

# 1 Introduction

Muth (1961, pp. 315-316) advanced the pathbreaking hypothesis that an economist can specify market participants' expectations of future outcomes as "essentially the same as the predictions of the relevant economic theory." According to Muth, a "relevant" model would enable "sensible predictions about the way expectations would change when either the amount of information or the structure of the [economic] system is changed" (pp. 315-316).

The rational expectations hypothesis (REH) implements Muth's hypothesis by supposing that the "relevant economic theory" of aggregate outcomes, such as inflation and unemployment, is a time-invariant macroeconomic model. Such a model characterizes how outcomes unfold over infinite past and indefinite future with an unchanging stochastic process.[1] REH represents participants' expectations of future outcomes with the conditional expectation implied by that process. Although behavioral models assume that participants' expectations deviate from REH, they also rest on the premise that aggregate outcomes can be characterized with a time-invariant model.

This paper proposes a novel approach to representing participants' expectations in macroeconomic models. We refer to our approach as the Knight-Muth hypothesis (KMH). Like REH, KMH maintains Muth's hypothesis. However, moving beyond REH, we implement Muth's hypothesis in a macroeconomic model that is open to unforeseeable change in the economy's structure. The defining feature of this change is that it is nonrecurring: the model's structure characterizing future outcomes differs in unforeseeable ways from that characterizing past outcomes. This implies that the timing and magnitude of future change cannot be known in advance even in probabilstic terms, which gives rise to Knightian uncertainty about future outcomes.

KMH builds on and provides empirical support for Knight's and Muth's profound insights. Knight (1921) argued that to understand profit-seeking activity in real-world markets, economists must recognize that they as well as market participants face unforeseeable change in the process driving outcomes. As he put it: "if all changes were to take place in accordance with invariable and universally known laws, [so that] they could be foreseen for an indefinite period in advance of their occurence (...) profit or loss would not arise" (Knight, 1921, p. 198). Such change, Knight argued, gives rise to "true uncertainty" (typically referred to today as Knightian uncertainty), which, in contrast to probabilistic risk, cannot be characterized *ex ante* in probabilistic terms.

Muth's (1961, pp. 315-316) idea was that his hypothesis would enable "sensible" predictions about how participants would revise their expectations if the structure of the economic system changes. KMH formalizes this idea when the process driving

[1]The structure of most time-invariant models – the set of explanatory variables, their parameters, and the probability distribution of the error term – is constrained to remain unchanging over time. Some models have allowed the structure to change over time by characterizing the parameters with a stationary Markov chain. However, these models are also time-invariant, in the sense that their structure is constrained to switch between the same recurring regimes according to a probabilistic rule, which enables them to represent participants' expectations with REH.

outcomes undergoes unforeseeable change.

To motivate our formalization of unforeseeable change, we show empirically that a first-order autoregressive process for U.S. inflation undergoes multiple nonrecurring structural shifts in the parameters characterizing the persistence and level of inflation.[2] We do so using quarterly inflation in the GDP deflator for the United States covering the sample period from 1969:Q1 to 2022:Q1. We find that the most and largest structural shifts occurred during the transitional period with high inflation in the 1970s and early 1980s. After the Volcker disinflation period, the parameters stabilize during the Great Moderation, with inflation fluctuating around a level of 2.5 percent until 2007 and 1.7 percent after 2007. At the end of the sample, we find evidence of structural shifts as inflation rises rapidly from 2020:Q3.

These empirical findings suggest that the "relevant economic model" to underpin Muth's hypothesis should allow for unforeseeable change in its parameters. Consequently, we present KMH in a baseline New Keynesian Phillips curve (NKPC). We open the model to unforeseeable change by allowing the parameters formalizing the level of inflation and the persistence of an output gap to undergo nonrecurring shifts. Importantly, because the timing and magnitude of these structural shifts cannot be known in advance, KMH implies that, viewed from any point in time, the conditional distribution that characterizes future inflation is unknown.

However, by constraining the model's unknown future parameters to lie within an interval at all times, we show that Muth's hypothesis represents the model's predictions of future inflation with a set of conditional distributions. Each of these distributions is indexed by a unique sequence of the unknown future parameters within the model's interval. At each point in time, KMH represents participants' expectation of future inflation with one of the conditional distributions in this set.

KMH acknowledges that participants revise their expectations in anticipation of and response to nonrecurring structural shifts in the inflation process. To formalize this, we allow different conditional distributions within the set constituting the model's predictions to represent participants' inflation expectation during different subperiods of time. In contrast, REH represents participants' inflation expectation at all times with the same conditional distribution, which is a time-invariant model's *only* prediction of future outcomes.

We derive the NKPC model's reduced form for inflation and show that it implies that both inflation and inflation expectations can be characterized with a first-order autoregressive process whose parameters undergo structural shifts. The former implication regarding inflation is consistent with the empirical results summarized above.

Using survey data on inflation forecasts we present empirical evidence for the prediction for inflation expectations: we find multiple structural shifts in a first-order au-

[2]Several studies testing theoretical representations of participants' inflation expectations using survey forecast data have used a first-order autoregressive process to characterize how inflation unfolds over time. These include the noisy-information representations in Coibion and Gorodnichenko (2015) and Angeletos, Huo, and Sastry (2021), and the behavioral representation based on Diagnostic Expectations in Bordalo et al. (2020).

toregressive process for the survey forecasts. To do so, we use the one-quarter-ahead forecast of GDP deflator inflation from the Survey of Professional Forecasters. As with the inflation data, we find that the most and largest structural shifts occured during the transitional period of high inflation in the 1970s, and specifically during the Volcker disinflation period of the early 1980s. After that, the parameters stabilize and the survey inflation forecasts fluctuate around a level of 2.1 during the Great Moderation. This level is slightly lower than the estimated level of inflation until 2007 and slightly higher than that level after 2007.

These empirical results provide support for a KMH's key prediction: the parameter estimates and the structural shifts in the autoregressive processes for inflation and inflation survey forecasts should broadly match, albeit *imprecisely.* We find that parameter estimates co-move over time. In particular, the estimated levels of inflation and inflation survey forecasts co-move over the sample period and were almost identical during the Great Moderation, when inflation expectations were well anchored suggesting KMH's potential usefulness in policy analysis. We also find that the frequent and large shifts in the autoregressive process for inflation during the 1970s and early 1980s are broadly matched by frequent and large shifts in the autoregressive process for the inflation survey forecasts.

In an important paper, Coibion and Gorodnichenko (2015) showed that full-information rational expectations (FIRE) are inconsistent with the survey data on inflation forecasts. Coibion, Gorodnichenko, and Kamdar (2018, pp. 1451-52) survey the "vast literature that tests the null hypothesis of FIRE" in the context of inflation forecasts and conclude that this literature "consistently finds that survey-based expectations deviate from FIRE."

KMH and our empirical results provide a novel explanation of these empirical difficulties: a "relevant economic theory" underpinning Muth's representation of expectations should allow for unforeseeable change in the process driving aggregate outcomes, such as inflation. Indeed, if the process driving inflation undergoes unforeseeable change, one would expect profit-seeking, forward-looking market participants' to recognize that they face such change, thereby forming expectations that deviate from FIRE.

Behavioral models imply that the influence of psychological factors lead market participants to commit predictable forecast errors. Gennaioli and Shleifer (2018) and Bordalo *et al.*'s (2020) formalization of diagnostic expectations provide a recent example of such a prediction, which they called "overreaction" in macroeconomic expectations.

To be sure, the evidence that psychological factors, such as market sentiment, influence participants' expectations is compelling. However, behavioral models assume that aggregate outcomes and participants' expectations can be characterized with time-invariant stochastic processes. Because REH rules out the influence of psychological factors *by design*, formalizing the influence of such factors has required behavioral models to represent participants' expectations as deviations from REH. In contrast, acknowledging that economists and market participants face Knightian uncertainty arising from unforeseeable change enables KMH to reconcile participants' reliance on psychological factors with Muth's hypothesis.

In REH models, participants' expectations of future outcomes are *fully* determined by the model's specification of these outcomes. Therefore, participants' expectations do not play an autonomous role in driving aggregate outcomes. In contrast, KMH implies such an autonomous role: although the model's specification of outcomes determines the set of model-consistent expectations of these outcomes, it does not determine which of the conditional expectations in this set represents participants' expectations.

In his landmark microfoundations volume, Phelps (1970, p. 22) conjectured that allowing participants' expectations to play an autonomous role in an economist's model would be crucial for understanding how profit-seeking, forward-looking market participants form expectations of inflation and how inflation unfolds over time. However, because the models presented in the Phelps volume were time-invariant, they had to rely on model-inconsistent specifications, such as adaptive expectations, to recognize an autonomous role for participants' expectations in driving inflation.[3] KMH provides a formalization of Phelps' conjecture that maintains Muth's hypothesis.

Our empirical findings suggest that acknowledging unforeseeable change and Knightian uncertainty that such change engenders in an otherwise standard macroeconomic model appears to be key to understanding how inflation and participants expectations of inflation evolve over time.

The plan of the paper is as follows. Section 2 presents evidence of structural shifts in an autoregressive process for inflation. Section 3 opens a baseline NKPC model to nonrecurring shifts in its parameters and develops KMH's representation of participants' expectation of inflation undergoing such shifts. Section 4 tests KMH's main prediction with survey data on participants' inflation forecasts. Section 5 concludes the paper with remarks on how KMH relates to, and moves beyond, the literature recognizing that market participants face ambiguity about the process driving outcomes.

# 2 Evidence of Structural Shifts in an Autoregressive Process for Inflation

Several studies testing theoretical representations of inflation expectations with survey data illustrate their approach in the context of a simple, typically first-order, autoregressive specification with constant parameters. This specification abstracts from the endogeneity of inflation and its dependence on participants' expectations of future inflation, as typically formalized in standard New Keynesian Phillips curve (NKPC) models. However, as we illustrate in the next Section, the characterization of inflation with the first-order autoregressive process can be derived from an NKPC models with parameters constrained to be constant over time.

[3] Frydman and Phelps (2013, p. 7) discuss this point in the context of macroeconomic models of the 1960s. For a penetrating critique of these models, and more broadly, intertemporal macroeconomic models relying on model-inconsistent specifications of participants' expectations, see Lucas (1995, p. 254 and 2005, p. 283).

Characterizing inflation with a simple autoregressive process has facilitated the derivation of alternative representations of participants' expectations of inflation and testing their implications against survey data. For example, Coibion and Gorodnichenko (2015) showed that professional forecasters' aggregate *ex post* inflation forecast errors, measured using survey data of inflation forecasts from the Survey of Professional Forecasters, are correlated with *ex ante* forecast revisions. This empirical finding, replicated by many other studies, rejects full-information rational expectations' (FIRE) implication that forecast errors should not be predictable.

However, using an autoregressive process to characterize how inflation unfolds over time, Coibion and Gorodnichenko illustrate that predictable forecast errors can occur if participants face noisy information and their expectations are represented with limited information rational expectations (LIRE). Angeletos, Huo, and Sastry (2021) extend Coibion and Gorodnichenko's noisy-information representation of expectations by adding misspecified beliefs, while Bordalo *et al.* (2020) develop a behavioral representation of participants' expectations based on diagnostic expectations.

Although these representations of participants' expectations differ in important aspects, they share the premise that inflation can be characterized with a time-invariant autoregressive process. Here, we examine empirically whether the parameters of such a process have undergone structural shifts or remained constant over time.

To do so, we consider a first-order autoregressive process with structural shifts at times $\{T_j\}_{j=1}^{K}$, which we specify as follows,

$$\pi_t = \rho^j \pi_{t-1} + \mu^j + \varepsilon_t, \tag{1}$$

for $t = T_{j-1}, T_{j-1}+1, \ldots, T_j - 1$ and $j = 1, 2, \ldots, K$, where $T_0 = 1$, $T_K = T + 1$, either $\rho^j \neq \rho^{j-1}$ or $\mu^j \neq \mu^{j-1}$ (such that at least one of the two parameters changes from subperiod $j-1$ to $j$) and $0 \leq \rho^j < 1$ for all $j$, $\varepsilon_t \sim iidN\left(0, \sigma^2\right)$, and the initial value $\pi_0$ is given. During subperiod $j$, the model's parameters are $(\rho^j, \mu^j)$ with the inflation persistence determined by $\rho^j$ and inflation fluctuating around a level of $\mu^j / \left(1 - \rho^j\right)$.

We use an effective estimation sample with 213 quarterly observations of U.S. inflation from 1969:Q1 to 2021:Q4. Following the practice in the literature, we use quarterly observations of the annualized change in the price level of the gross domestic product (PGDP). This matches the measure of inflation and the sample period for which the survey of inflation forecasts is available. A full description of the data is given in Online Appendix B.1.

We estimate the autoregressive process in (1) both without structural shifts, i.e. with $\rho^j = \rho$ and $\mu^j = \mu$ for all $t = 1, 2, \ldots, T$, and with structural shifts in $(\rho^j, \mu^j)$. To identify the timing of shifts $\{T_j\}_{j=1}^{K}$ and estimate the parameters $\{\rho^j, \mu^j\}_{j=1}^{K}$ from the time-series data, we use the Autometrics tree-search algorithm with step-indicator and multiple step-indicator saturation, as sketched in Online Appendix B.2.[4] This procedure has two important advantages: it allows for shifts in $\rho^j$ and $\mu^j$ to occur at

[4]For a presentation of Autometrics and an analysis of its properties, see Doornik (2009), Castle *et al.* (2012), and Castle *et al.* (2015).

Table 1: Estimates of the Autoregressive Process for Inflation with Constant Parameters and with Structural Shifts in the Parameters

| Model with constant parameters | | | Model with structural shifts | | | |
|---|---|---|---|---|---|---|
| Parameter | Estimate | Std. error | Parameter, period | | Estimate | Std. error |
| $\rho$ | 0.895 | 0.032 | $\rho^j$ | 69:1-72:2 | -0.033 | 0.118 |
| | | | | 72:3-74:2 | 0.755 | 0.079 |
| | | | | 74:3-74:3 | 1.150 | 0.098 |
| | | | | 74:4-74:4 | 0.803 | 0.069 |
| | | | | 75:1-20:1 | 0.525 | 0.047 |
| | | | | 20:2-20:3 | -1.649 | 0.363 |
| | | | | 20:4-20:4 | 0.393 | 0.222 |
| | | | | 21:1-22:1 | 1.034 | 0.070 |
| $\mu$ | 0.380 | 0.136 | $\mu^j$ | 69:1-72:1 | 5.262 | 0.635 |
| | | | | 72:2-76:3 | 2.329 | 0.410 |
| | | | | 76:4-79:4 | 3.628 | 0.409 |
| | | | | 80:1-81:1 | 5.138 | 0.568 |
| | | | | 81:2-82:4 | 2.440 | 0.456 |
| | | | | 83:1-06:4 | 1.190 | 0.146 |
| | | | | 07:1-07:1 | 3.262 | 0.801 |
| | | | | 07:2-22:1 | 0.786 | 0.132 |
| $\sigma$ | | 1.158 | | | | 0.798 |
| $R^2$ | | 0.79 | | | | 0.91 |
| Observations | | 213 | | | | 213 |
| Misspecification tests | | [p-value] | | | | [p-value] |
| No autocorr., order 1-2 | | [0.001] | | | | [0.159] |
| No ARCH, order 1-4 | | [0.002] | | | | [0.655] |
| No heteroskedasticity | | [0.000] | | | | [0.331] |
| Normality | | [0.000] | | | | [0.372] |

Notes: The table shows estimates of the autoregressive process for inflation in (1) with constant parameters and with structural shifts in the parameters. The effective estimation sample is quarterly observations from 1969:Q1 to 2022Q:1.

any time during the sample period, and it allows the two parameters to shift at different times.

The empirical estimates of the autoregressive process in (1) with constant parameters are shown in the left columns of Table 1. Figure 1 shows the actual inflation data and the constant-parameter process's fitted inflation $\pi_t - \hat{\varepsilon}_t$ in panel (a), the standardized estimated residuals in panel (b), and 10-year rolling window estimates of $\rho$ and $\mu$, respecitively, in panels (c) and (d).

Constraining parameters to be constant over time, we find that the estimated persistence of inflation is high at 0.895 (std. error of 0.032), and, with the estimate of $\mu$ equal to 0.380 (std. error of 0.136), inflation fluctuates around a long-run level of 3.6 percent. However, the misspecification tests strongly reject the null hypotheses of no autocorrelation, no ARCH, no heteroskedasticity, and normality of the residuals. Thus, the autoregressive process in (1) with constant parameters is not an adequate representation of the inflation data.

Although this misspecification can be caused by many factors, the rolling window estimates in panels (c) and (d) of Figure 1 provide one potential reason: the estimated parameters do not appear constant over the sample period. Indeed, considering a variety of inflation data and measurements of persistence, Fuhrer's (2010, p. 449) survey of inflation persistence concludes that "[w]eighing all of the evidence, it seems reasonable to conclude that the persistence of inflation has decreased somewhat in recent years."

The right columns of Table 1 show the estimates of the autoregressive process in (1) with structural shifts in the parameters $(\rho^j, \mu^j)$. Figure 2 shows the actual inflation data and the fitted inflation $\pi_t - \hat{\varepsilon}_t$ from this process in panel (a), the standardized estimated residuals in panel (b), and the estimates of $\rho^j$ and $\mu^j$ over time, respectively, in panels (c) and (d).

We find evidence of multiple shifts in both parameters. All shifts are statistically significant and, importantly, they appear to be nonrecurring rather than shifting between a fixed set of, say two, possible values. As all misspecification tests do not reject the null hypotheses with p-values well above 5 percent, we conclude that allowing for shifts in the parameters renders the autoregressive process an adequate representation of the time-series data on inflation.

Although the shifts in $\rho^j$ and $\mu^j$ occur at different points in time, we find that the most and largest shifts occurred during the transitional period of high inflation in the 1970s and rapid disinflation in the early 1980s. As Figure 3 illustrates, these changes lead to frequent and large shifts in the level, $\mu^j/(1-\rho^j)$, that inflation fluctuates around during the 1970s.[5]

Allowing for structural shifts yields estimates that are very different from the constant-parameter estimates. In particular, when the structural shifts are accounted for, the estimated inflation persistence drops from 0.895 in the time-invariant model to lower values throughout the majority of the sample period.

[5] In Appendix A.1, we show that we still find structural shifts in $(\rho^j, \mu^j)$ when the autoregressive process with structural shifts in (1) includes four lags of inflation.

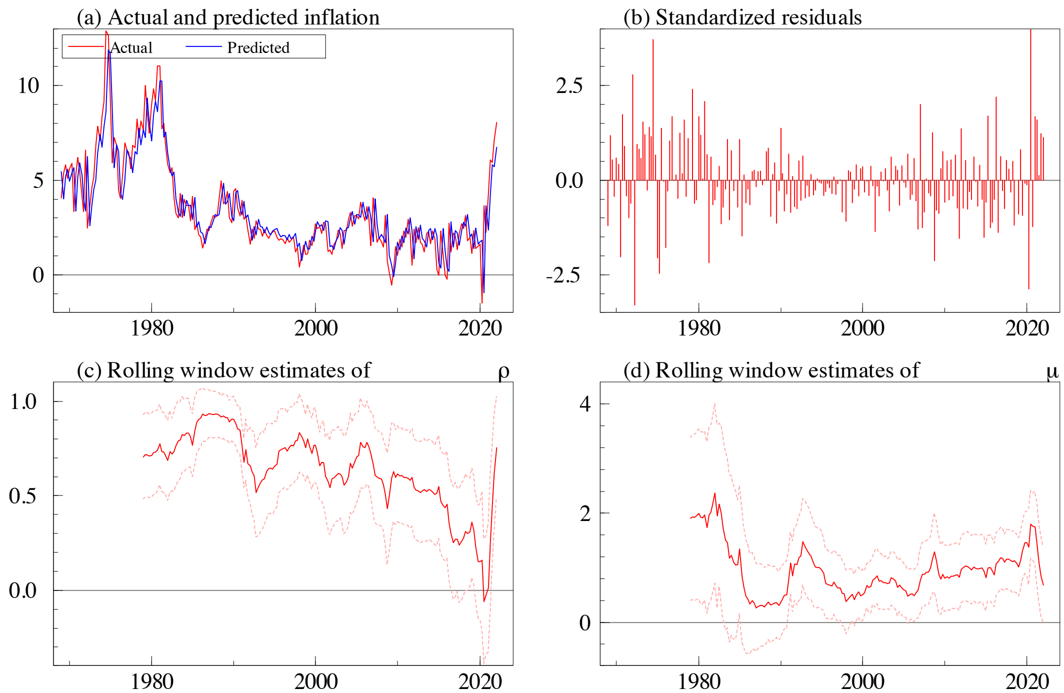


Figure 1: The plot illustrates the estimated autoregressive process for inflation $\pi_t$ in (1). Panel (a) shows the actual inflation $\pi_t$ (red line) and the model's predicted inflation $\hat{\pi}_t = \pi_t - \hat{\varepsilon}_t$ (blue line). Panel (b) shows the standardized estimated residuals. Panels (c) and (d) show the rolling window estimates of the parameters $\rho$ and $\mu$ (red lines) and their 95 percent confidence intervals (dotted red lines) based on a 10-year rolling window sample that ends at the point in time illustrated by the red lines.

In the first years of the sample, we find that the estimated persistence $\rho^j$ is insignificantly different from zero. But it increases to 0.755 (std. error of 0.079) in 1972:Q3 and momentarilty increases further to 1.150 (std. error of 0.098) in 1974:Q3. Estimated persistence drops to 0.803 (std. error of 0.069) in 1974:Q4 and to 0.525 (std. error of 0.047) in 1975:Q1. It remains at that level until 2020. As inflation rises rapidly in 2020:Q2, the estimated persistence increases to 1.034 (std. error of 0.070), such that the inflation process essentially becomes a random walk with a drift or even momentarily an explosive process in violation of the assumption of (1) that $\rho^j < 1$.

The decline in inflation persistence over the sample period is consistent with the empirical results in the literature, as surveyed by Fuhrer (2010). For example, Pivetta and Reis (2007) estimate the first-order autocorrelation of U.S. inflation using rolling window samples (Fuhrer, 2010, extends their sample period until 2010) and find it to be 0.8 from the 1970s until the mid-1990s, when it drops to a range of 0.5 − 0.6 and

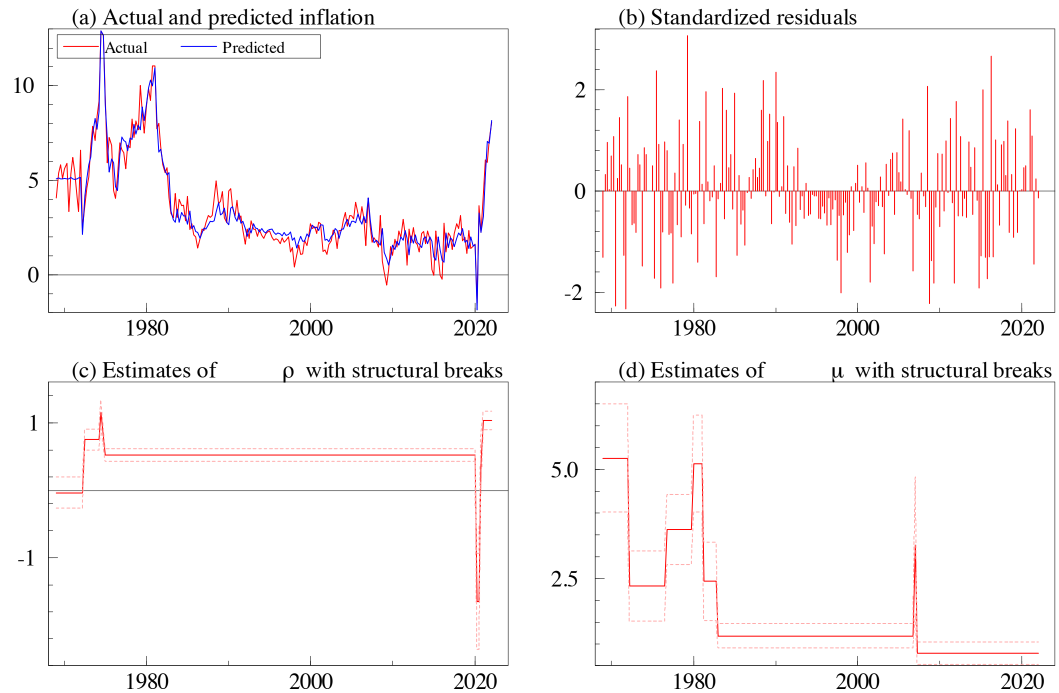


Figure 2: The plot illustrates the estimated autoregressive process with structural breaks for inflation $\pi_t$ in (1). Panel (a) shows the actual inflation $\pi_t$ (red line) and the model's predicted inflation $\hat{\pi}_t = \pi_t - \hat{\varepsilon}_t$ (blue line). Panel (b) shows the standardized estimated residuals. Panels (c) and (d) show the estimates of $\rho^j$ and $\mu^j$ with structural breaks (red lines) and their 95 percent confidence intervals (dotted red lines).

then to a range of $0.0 - 0.4$ in the mid-2000s.[6] These ranges broadly correspond to our estimates of $\rho^j$. However, our findings suggest that once the substantial structural shifts in the parameter $\mu^j$ are taken into account, the estimated inflation persistence drops to around 0.5 already in the mid-1970s and remains at this level until the rapid rise in inflation in 2020.

During the initial years of the sample before 1972:Q1, the estimate of $\mu^j$ is high at 5.262 (std. error of 0.635). As the persistence was insignificantly different from zero, this implies that inflation fluctuates around this level during this period, as illustrated in Figure 3. The high estimated persistence from 1972 is matched by a decrease in the estimate of $\mu^j$ to 2.329 (std. error of 0.410) in 1972:Q2, such that the large and

[6] Fuhrer (2010) also estimates a univariate autoregressive process for detrended inflation with structural shifts identified by the tests for unknown breakpoints by Andrews (1993) and Bai and Perron (1998). He considers different inflation measures, detrended by a Hodrick-Prescott filter to remove a time-varying mean, and finds a significant structural break in inflation persistence occuring in 1980:Q2 for core CPI inflation, another in 1976:Q2 for core PCE inflation, and one in 1999:Q1 for CPI and PCE inflation.

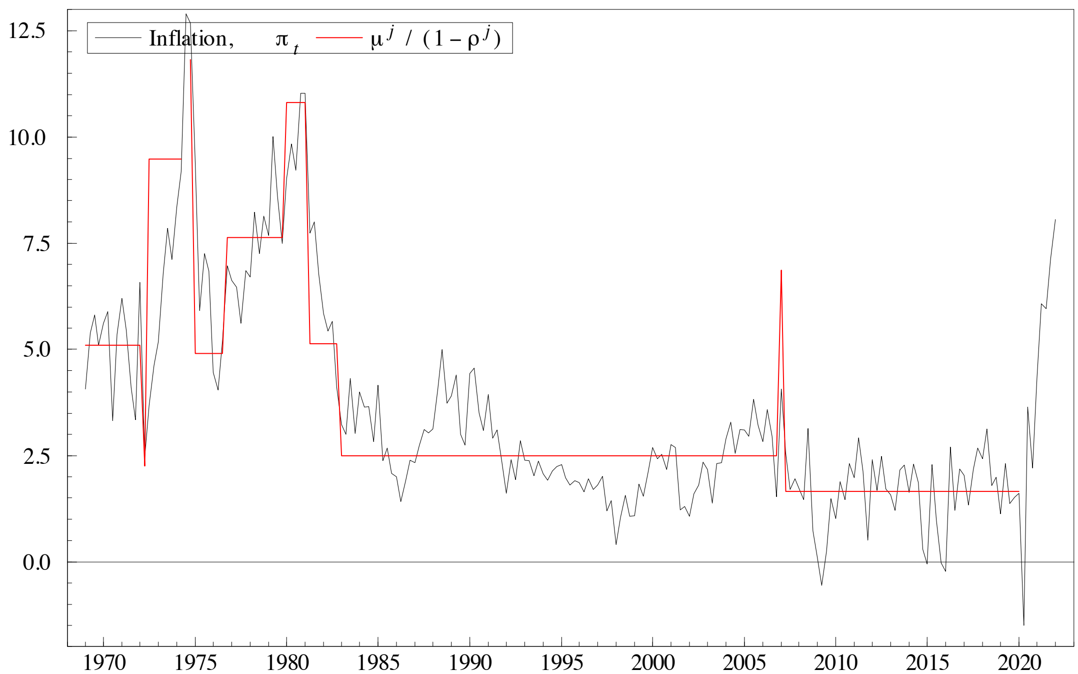


Figure 3: The figure shows the actual inflation $\pi_t$ (black line) and the estimated level of inflation $\mu^j/(1-\rho^j)$ from (1) (red line). Because the interpretation of $\mu^j/(1-\rho^j)$ as the level for inflation requires that $|\rho^j| < 1$, the red line excludes the eight observations where the estimates of $\rho^j$ are either smaller than -1 or larger than 1.

frequent shifts in the level of inflation from 1972 to 1974 are caused by shifts in $\rho^j$. As the estimated persistence stabilizes from 1975, the increasing inflation in the last part of the 1970s and the rapid decrease in inflation during the Volcker disinflation period of the early 1980s are captured by an increase followed by a decrease in the estimates of $\mu^j$. With the stabilization of inflation from 1983 and during the Great Moderation, the estimate of $\mu^j$ stays at 1.190 (std. error of 0.146) from 1983:Q1 until 2006:Q4, when $\mu^j$ temporarily increases and then settles at 0.786 (std. error of 0.132) in 2007:Q2 until the end of the sample.

Our empirical findings show that the parameters of a first-order autoregressive process undergo nonrecurring structural shifts in its parameters and that it provides an adequate characterization of the inflation data once these shifts are accounted for. This suggests that the "relevant economic model" to underpin Muth's hypothesis should allow for such nonrecurring structural shifts in its parameters. Given these findings, the rest of the paper develops the Knight-Muth hypothesis in a baseline New Keynesian Phillips curve model. We show that this implies that inflation can be characterized by a first-order autoregressive process with nonrecurring structural shifts in its parameters, in line with our empirical results, and we explore the possibility that KMH's implementation of Muth's hypothesis under Knightian uncertainty could account for market participants' inflation expectations, as measured by the survey data.

# 3 The Knight-Muth Hypothesis in a Baseline New Keynesian Phillips Curve Model

Our implementation of the Knight-Muth hypothesis in a baseline New Keynesian Phillips curve (NKPC) model of inflation involves two steps.[7] First, we open the model to unforeseeable change and Knightian uncertainty on the part of the economist and market participants by allowing some of its parameters to undergo nonrecurring structural shifts. Second, we show that under Knightian uncertainty, Muth's hypothesis implies that a set of conditional expectations constitutes the model's prediction of future inflation. We implement Muth's hypothesis by representing participants' inflation expectation with one of the conditional expectations in this set.

KMH thus moves beyond REH by implementing Muth's hypothesis in a macroeconomic model open to unforeseeable change and Knightian uncertainty, though it nests REH as a special case. We illustrate this by showing how KMH reduces to REH when the parameters of the NKPC model are constrained to remain constant over time. This enables a direct comparison of the different implications of KMH and REH for understanding how market participants' expectations evolve over time and what role these expectations play in driving aggregate outcomes resulting from participants' decisions.

## 3.1 Opening the Model to Knightian Uncertainty Arising from Unforeseeable Change

We consider a baseline NKPC model of inflation $\pi_t$ specified as:

$$\pi_t = \bar{\omega}_t + \beta \left( F_t \left( \pi_{t+1} \right) - \bar{\omega}_t \right) + \kappa y_t, \tag{2}$$

for $t = 1, 2, \ldots$ and where $\bar{\omega}_t$ is a parameter at time $t$, which denotes a non-zero level of inflation; $(\beta, \kappa)$ are parameters that are constant over time; $F_t\left(\pi_{t+1}\right)$ denotes market participants' expectation at time $t$ of the next period's inflation; and $y_t$ represents an output gap, as measured by the deviation of the economy's real output, unemployment, or firms' marginal cost from their respective trend levels.

We specify the output gap to evolve according to the first-order autoregressive process:

$$y_t = \bar{\phi}_t y_{t-1} + \eta_t, \tag{3}$$

[7] Coibion, Gorodnichenko, and Kamdar (2018) provide an overview and historical account of the role of expectations and their different theoretical representations in the context of New Keynesian Phillips curve models.

for $t = 1, 2, \ldots$ and where $\eta_t \sim iidN\left(0, \sigma_\eta^2\right)$, $\bar{\phi}_t$ is an autoregressive parameter at time $t$ that satisfies $0 \leq \bar{\phi}_t < 1$ for all $t$, and the initial value $y_0$ is given.

The simple structure of this model corresponds to the "baseline case" of an NKPC model in Fuhrer (2010), but with a non-zero level of inflation.[8]

The premise of our approach is that some parameters of the NKPC model in (2)-(3) undergo unforeseeable change. Here, we focus on the shifts in the level of inflation, $\bar{\omega}_t$, and the autoregressive parameter, $\bar{\phi}_t$, for the output gap.

Let $\bar{\theta}_t = \left(\bar{\omega}_t, \bar{\phi}_t\right)'$ denote the two-dimensional vector of parameters at time $t$ in (2)-(3). We refer to the *particular* deterministic sequence $\left\{\bar{\theta}_t\right\}_{t=1}^{\infty}$ as the objective Knightian uncertainty (KU) parameters.

We formalize unforeseeable change in the model by allowing the objective KU parameters to undergo nonrecurring shifts over time. However, we assume that there are subperiods during which $\bar{\theta}_t$ can be represented as constant, which we specify with

$$\bar{\theta}_t = \left(\bar{\omega}_t, \bar{\phi}_t\right)' = \bar{\theta}^j = \left(\bar{\omega}^j, \bar{\phi}^j\right)', \tag{4}$$

for $t = \bar{\tau}_{j-1}, \bar{\tau}_{j-1} + 1, \ldots, \bar{\tau}_j - 1$ and where $j = 1, 2, \ldots$ denotes the different subperiods with $\bar{\tau}_j > \bar{\tau}_{j-1}$ and $\bar{\tau}_0 = 1$. We assume that either $\bar{\omega}^j \neq \bar{\omega}^{j-1}$, $\bar{\phi}^j \neq \bar{\phi}^{j-1}$, or both, such that at least one of the two objective KU parameters changes from one subperiod to the next. This formalization allows the objective KU parameters to change only intermittently: they are constant at all times during subperiod $j$, where $\bar{\theta}_t = \bar{\theta}^j$, but they undergo nonrecurring structural shifts at times $\{\bar{\tau}_1, \bar{\tau}_2, \ldots\}$. Importantly, we assume that the timing and magnitude of *future* shifts are inherently unknown *ex ante*: at any time $t$, the future objective KU parameters $\bar{\theta}_{t+i}$, $i > 0$, cannot be assessed on the basis of historical data (even an infinite sample), let alone characterized with a probabilistic rule, such as a Markov chain.[9]

For the model to yield predictions about future inflation and the output gap, we restrict the objective KU parameters to lie within an interval at all times. Assuming that this interval is constant over time, we formalize this restriction on $\bar{\theta}_t$ as

$$\bar{\theta}^j \in I^\theta = I^\omega \times I^\phi, \qquad \text{for all } j = 1, 2, \ldots, \tag{5}$$

where the intervals $I^\omega$ and $I^\phi$ are given by

$$I^\omega = [\omega_L, \omega_U], \qquad 0 < \omega_L < \omega_U, \qquad I^\phi = [\phi_L, \phi_U], \qquad 0 \leq \phi_L < \phi_U < 1. \tag{6}$$

Because $\bar{\theta}_t \in I^\theta$ for all $t$, this restriction limits the extent of change that can occur in the model's objective KU parameters. However, it still leaves the model open to unforeseeable change and Knightian uncertainty, in the sense that, as viewed from time $t$, the

[8] Kozicki and Tingsley (2002), Cogley and Sbordone (2008), and subsequent papers also allow for a non-zero steady-state level of inflation.

[9] The structural shifts that have occured in the past, their timings, and the parameter values can be estimated on the basis of a sample of time-series data.

objective future KU parameters can take any value within the interval. Specifically, let $\bar{\Theta}_t = \{\bar{\theta}_{t+1}, \bar{\theta}_{t+2}, \ldots\}$ denote the particular sequence of objective KU parameters that characterizes future inflation and the output gap, $\{\pi_{t+i}, y_{t+i}\}_{i=1}^{\infty}$, according to (2)-(3). Viewed from time $t$, $\bar{\Theta}_t$ can take any value within the interval $I^{\theta} \times I^{\theta} \times \ldots$.

## 3.2 Implementing Muth's Hypothesis Under Knightian Uncertainty

Lucas (1995, p. 254-255) pointed out that intertemporal macroeconomic models that do not represent participants' expectations as conforming to Muth's hypothesis suffer from a "*glaring* (...) modeling inconsistency" (emphasis added). Such models' characterization of the "actual equilibrium prices (...) bore no relation to, and [are] in general grossly inconsistent with, the price expectation that the theory imputed to individual agents."[10]

By implementing Muth's hypothesis, REH rids macroeconomic theory of this modeling inconsistency. Muth's (1961, pp. 315-316) pathbreaking idea was to determine simultaneously the objective stochastic process which, according to an economist's model, characterizes time-series inflation data and the inflation expectations driving participants' decisions that result in that time series. As Lucas emphasized in the panel discussion in Hoover and Young (2013, p. 1172), "It's that simultaneity that neither the statistician nor the economist of the day had (...) and Jack [Muth] had it." This simultaneity renders REH's representation of participants' subjective expectation consistent with the model's objective prediction of future inflation, in the sense that the conditional expectation that represents participants' subjective expectation of inflation is identical to the model's objective conditional expectation of inflation.

Similarly, KMH ensures such modeling consistency by relying on Muth's hypothesis to specify simulataneously the model-implied inflation process and participants' subjective expectation of inflation. However, implementing Muth's hypothesis under Knightian uncertainty substantially alters what constitutes the model's objective prediction of inflation and, thus, how KMH represents participants' subjective expectation of inflation as being consistent with that prediction.

### 3.2.1 The Model's Prediction of Future Inflation Under Knightian Uncertainty

Because the KMH model is open to unforeseeable change, neither an economist nor market participants can know *ex ante* which particular sequence of objective KU parameters within the model's interval will characterize how inflation and the output gap unfold in the future. However, an extension of the standard REH solution method enables us to use Muth's hypothesis to derive an expectation of $\pi_{t+1}$, conditional on

[10]Lucas (1995, p. 254) criticized the models of the 1960s that typically represented participants' expectations with adaptive expectations and other fixed error-correcting rules. However, his argument also applies to behavioral macroeconomic models.

the information set at time $t$, that is consistent with any sequence of future parameters within the model's interval.

To this end, we denote an arbitrary sequence of the model's future parameters with

$$\Theta_t = \{\theta_{t,t+1}, \theta_{t,t+2}, \ldots\}, \qquad \text{where } \theta_{t,t+i} = \left(\omega_{t,t+i}, \phi_{t,t+i}\right)' \in I^{\theta} \text{ for all } i > 0.$$

We refer to $\Theta_t$ as a scenario, as viewed from a *fixed* time $t$.

Given a particular scenario, Muth's hypothesis determines the process for future inflation and the output gap simultaneously with participants' inflation expectation. Future inflation and the output gap will evolve according to the process in (2)-(3), but with the unknown future objective KU parameters, $\bar{\Theta}_t$, replaced by to the scenario's parameters, $\Theta_t$. We denote this process by $\{\pi_{t+i}(\Theta_t), y_{t+i}(\Theta_t)\}_{i=1}^{\infty}$. Muth's hypothesis also implies that the representation of participants' inflation expectation that is consistent with this scenario is given by $F_{t+i}(\pi_{t+i+1}(\Theta_t)) = E_{t+i}(\pi_{t+i+1}(\Theta_t))$ for all $i > 0$. Here, $E_t(\pi_{t+i+1}(\Theta_t))$ denotes the conditional expectation implied by the scenario's process $\{\pi_{t+i}(\Theta_t), y_{t+i}(\Theta_t)\}_{i=1}^{\infty}$, given by:

$$\pi_{t+i}(\Theta_t) = \omega_{t,t+i} + \beta\left(E_{t+i}(\pi_{t+i+1}(\Theta_t)) - \omega_{t,t+i}\right) + \kappa y_{t+i}(\Theta_t), \tag{7}$$

$$y_{t+i}(\Theta_t) = \phi_{t,t+i} y_{t+i-1}(\Theta_t) + \eta_{t+i}, \tag{8}$$

for $i = 1, 2, \ldots$. We refer to this expectation and process as scenario-consistent.

Any scenario-consistent representation of participants' forecast of next-period inflation in the NKPC model in (2) is given by $F_t(\pi_{t+1}(\Theta_t)) = E_t(\pi_{t+1}(\Theta_t))$. An expression for this expectation involving $E_t(\pi_{t+n+1}(\Theta_t))$ can be derived by iterating it forward with respect to the process for $\{\pi_{t+i}(\Theta_t), y_{t+i}(\Theta_t)\}_{i=1}^{\infty}$ in (7)-(8) and using the law of iterated expectations. The following Proposition shows that $E_t(\pi_{t+n+1}(\Theta_t))$ converges as $n \to \infty$ and derives an explicit solution for the scenario-consistent expectation $E_t(\pi_{t+1}(\Theta_t))$ for $n \to \infty$, assuming that a standard transversality condition holds. This results from the scenario's parameters being bounded by the model's interval $I^{\theta}$.

**Proposition 1** *Let $\Theta_t = \{\theta_{t,t+1}, \theta_{t,t+2}, \ldots\}$ with $\theta_{t,t+i} = \left(\omega_{t,t+i}, \phi_{t,t+i}\right)' \in I^{\theta}$ denote a scenario for the future parameters within the model's interval $I^{\theta}$, as viewed from time $t$, which implies that $0 < \omega_L \leq \omega_{t,t+i} \leq \omega_U$ and $0 \leq \phi_L \leq \phi_{t,t+i} \leq \phi_U < 1$ for all $i > 0$, see (6). Moreover, given the scenario $\Theta_t$, let $E_t(\pi_{t+1}(\Theta_t))$ denote the scenario-consistent expectation of $\pi_{t+1}(\Theta_t)$, conditional on the information set at time $t$, with future inflation and the output gap evolving according to the process $\{\pi_{t+i}(\Theta_t), y_{t+i}(\Theta_t)\}_{i=1}^{\infty}$ in (7)-(8).*

*By forward iteration, the scenario-consistent expectation $E_t(\pi_{t+1}(\Theta_t))$ can be ex-*

*pressed as:*

$$E_t\left(\pi_{t+1}\left(\Theta_t\right)\right) = (1-\beta)\sum_{i=1}^{n}\beta^{i-1}\omega_{t,t+i} + \beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right) + \kappa\sum_{i=1}^{n}\beta^{i-1}E_t\left(y_{t+i}\left(\Theta_t\right)\right)$$
$$= (1-\beta)\sum_{i=1}^{n}\beta^{i-1}\omega_{t,t+i} + \beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right) + \kappa\sum_{i=1}^{n}\beta^{i-1}\left(\prod_{j=1}^{i}\phi_{t,t+j}\right)y_t, \tag{9}$$

*where* $\lim_{n\to\infty}\beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right)$ *exists.*

*If, moreover, this limit satisfies the transversality condition*

$$\lim_{n\to\infty}\beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right) = 0, \tag{10}$$

*then the scenario-consistent expectation* $E_t\left(\pi_{t+1}\left(\Theta_t\right)\right)$ *in (9) has the unique solution for* $n\to\infty$ *given by:*

$$E_t\left(\pi_{t+1}\left(\Theta_t\right)\right) = (1-\beta)\sum_{i=1}^{\infty}\beta^{i-1}\omega_{t,t+i} + \kappa\sum_{i=1}^{\infty}\beta^{i-1}E_t\left(y_{t+i}\left(\Theta_t\right)\right)$$
$$= (1-\beta)\sum_{i=1}^{\infty}\beta^{i-1}\omega_{t,t+i} + \kappa\sum_{i=1}^{\infty}\beta^{i-1}\left(\prod_{j=1}^{i}\phi_{t,t+j}\right)y_t. \tag{11}$$

The proof of the proposition is in Appendix A.2.

Viewed from any time $t$, any scenario $\Theta_t$ within the model's interval $I^\theta \times I^\theta \times \ldots$ might characterize how inflation and the output gap evolve in the future. Therefore, Muth's hypothesis under Knightian uncertainty implies that a *set* of conditional distributions constitutes the model's *objective prediction* of next-period inflation, as viewed from time $t$. The following Corollary formalizes this set and shows that it corresponds to an interval with endpoints that depend on the endpoints of the interval $I^\theta$.

**Corollary 1** *Let* $E_t\left(\pi_{t+1}\left(\Theta_t\right)\right)$ *denote the scenario-consistent expectation given the scenario* $\Theta_t$*, as given by (11), and let* $\mathcal{E}_t\left(\pi_{t+1}; I^\theta\right)$ *denote the model-implied set of scenario-consistent expectations, conditional on the information set at time* $t$*. This set is given by*

$$\mathcal{E}_t\left(\pi_{t+1}; I^\theta\right) = \left\{E_t\left(\pi_{t+1}\left(\Theta_t\right)\right) \mid \Theta_t \in I^\theta \times I^\theta \times \ldots\right\}$$
$$= \left[\omega_L + \frac{\kappa\phi_L}{1-\beta\phi_L}y_t, \omega_U + \frac{\kappa\phi_U}{1-\beta\phi_U}y_t\right]. \tag{12}$$

The proof of the corollary is in Appendix A.2.

The Corollary highlights the difference between the model's prediction of next-period inflation under KMH and REH. By opening the NKPC model to unforeseeable change and restricting the objective KU parameters to lie within an interval at all times, KMH's implementation of Muth's hypothesis implies that the set of scenario-consistent expectations in (12) constitutes the model's objective prediction of $\pi_{t+1}$, as viewed from time $t$.

If the unforeseeable change is replaced by the assumption that the model's objective parameters are constant, $\bar{\theta}_t = \bar{\theta} = \left(\bar{\omega}, \bar{\phi}\right)'$ for all $t$ and $I^\theta = \left[\bar{\theta}, \bar{\theta}\right] = \bar{\theta}$, the NKPC model in (2)-(3) becomes time-invariant. Because future outcomes are an exact probabilistic replica of the past, there is no uncertainty about the objective parameters in the future. They are identical to those characterizing the process for inflation in the past, $\bar{\Theta}_t = \bar{\Theta} = \left\{\bar{\theta}, \bar{\theta}, \ldots\right\}$ for all $t$. Consequently, KMH becomes identical with REH as the set in (12) collapses to the single conditional expectation, $\mathcal{E}_t\left(\pi_{t+1}; \bar{\theta}\right) = E_t\left(\pi_{t+1}\left(\bar{\Theta}\right)\right) = \bar{\omega} + \left(\kappa\bar{\phi}/\left(1-\beta\bar{\phi}\right)\right) y_t$. This expectation depends only on the constant objective parameters, $\bar{\theta}$, and it constitutes an REH model's *only* objective prediction of $\pi_{t+1}$. By representing participants' expectation of inflation with this conditional expectation, REH renders expectations consistent with the model's objective prediction.

Corollary 1 also highlights how opening the NKPC model to unforeseeable change implies Knightian uncertainty about future inflation and the output gap: the uncertainty about these outcomes cannot *ex ante* be "reduced to an objectively, quantitatively determinate probability" (Knight, 1921, pp. 231-232) by any method. Because the future objective KU parameters are inherently unknown at time $t$, it cannot be known *ex ante* which of the scenario-consistent expectations in the set $\mathcal{E}_t\left(\pi_{t+1}; I^\theta\right)$ will actually characterize future inflation. In addition to Knightian uncertainty, the random shocks $\eta_t$ formalize standard probabilistic risk. In REH models, the uncertainty about future outcomes is formalized only as probabilistic risk.[11]

#### 3.2.2 Model-Consistent Representation of Participants' Subjective Inflation Expectation

By opening the model to unforeseeable change, an economist acknowledges that there are myriad scenarios for the future objective KU parameters and, importantly, that it is inherently unknown *ex ante* when and how these parameters will shift in the future. However, because KMH constrains the parameters to lie within the interval $I^\theta$, the model-implied set of scenario-consistent expectations in the NKPC model in (2) is given by $\mathcal{E}_t\left(\pi_{t+1}; I^\theta\right)$ in (12).

To implement Muth's hypothesis under Knightian uncertainty, we represent participants' expectation of inflation, $F_t\left(\pi_{t+1}\right)$ in (2), with the scenario-consistent expectation corresponding to a particular scenario, which we refer to as participants' subjective scenario. Using a general notation, we define this scenario as

$$\tilde{\Theta}_t = \left\{\tilde{\theta}_{t,t+1}, \tilde{\theta}_{t,t+2}, \ldots\right\}, \qquad \text{where } \tilde{\theta}_{t,t+i} = \left(\tilde{\omega}_{t,t+i}, \tilde{\phi}_{t,t+i}\right)' \in I^\theta \text{ for all } i > 0. \quad (13)$$

[11] Corollary 1 illustrates how KMH formalizes Knightian uncertainty as ambiguity about the process and distribution characterizing future inflation and the output gap. In the Concluding Remarks, we discuss how KMH relates to the literature on model-ambiguity and how KMH can be combined with the multiple priors utility approach to decision-making under ambiguity.

The subjective scenario represents participants' subjective assessment of the unknown future objective KU parameters, as viewed from time $t$. Thus, in the NKPC model in (2), we represent

$$F_t\left(\pi_{t+1}\right) = E_t\left(\pi_{t+1}\left(\tilde{\Theta}_t\right)\right), \tag{14}$$

at each time $t$, where $\tilde{\Theta}_t \in I^\theta \times I^\theta \times \ldots$ implies that $E_t\left(\pi_{t+1}\left(\tilde{\Theta}_t\right)\right) \in \mathcal{E}_t\left(\pi_{t+1}; I^\theta\right)$, as defined in (12).

KMH acknowledges that, over time, participants revise their subjective scenario, and thus their expectation of inflation, in unforeseeable ways, in response to and anticipation of the nonrecurring shifts in the model's objective KU parameters. It is possible that there are subperiods during which the parameters of the subjective scenario can be constrained by some rule or procedure. However, the premise of KMH is that *any* such rule or procedure must be left open to unforeseeable change.

Because the subjective scenario represents a subjective assessment of the unknown future objective KU parameters, the scenario can be decomposed into an assessment of the parameters' current values and future change.[12] The assessment of the current values might be formalized with some procedure that depends on the past time-series data for inflation and the output gap implied by the model. That would formalize how participants revise their subjective scenario in response to past shifts in the objective KU parameters.

However, even with a perfect assessment of the past and current objective KU parameters, such that $\left\{\bar{\theta}_i\right\}_{i=1}^{t}$ would be known at time $t$, forward-looking participants understand that the subjective scenario also requires an assessment of the timing and magnitude of change in the future. No time-invariant rule or procedure can characterize exactly how participants make and revise this assessment at all times. Indeed, because participants facing Knightian uncertainty rely on a variety of factors, including psychological and other factors outside the model's specification in (2), they might, at least intermittently, revise their assessment of the timing and magnitude of future shifts in ways that cannot be known in advance, let alone characterized with a probabilistic rule.[13] Participants may revise their subjective scenario in unforeseeable ways, purely in anticipation of future change in the objective KU parameters.

Here, we impose simple constraints on $\tilde{\Theta}_t$ that mimic our formalization of nonrecurring shifts in the objective KU parameters. To do so, we assume that the subjective

[12] The subjective scenario enters $E_t\left(\pi_{t+1}\left(\tilde{\Theta}_t\right)\right)$ through the iteration of expectations. Thus, the scenario's parameters $\tilde{\theta}_{t,t+2}$ represent a subjective assessment, at time $t$, of participants' subjective assessment at time $t+1$ of the objective KU parameters at time $t+2$. However, if participants anticipate, at time $t$, that they will revise their assessment of $\bar{\theta}_{t+2}$ at time $t+1$, they will revise their time-$t$ assessment of $\bar{\theta}_{t+2}$ to reflect that anticipated future revision. Thus, the parameters of the subjective scenario can be interpreted as a subjective assessment of the future objective KU parameters.

[13] This opens the possibility of formalizing the influence of psychological factors, such as market sentiment, on participants' subjective scenario and thus their expectation of inflation. For example, one could assume that optimistic market sentiment would lead participants to shift the parameters of the subjective scenario upward.

parameters also undergo nonrecurring structural shifts, but that there are subperiods during which they remain constant. Moreover, we make the simplifying assumption that, during each subperiod, the subjective parameters are constant over the forecast horizon, $\tilde{\theta}_{t,t+i} = \tilde{\theta}_t = \left(\tilde{\omega}_t, \tilde{\phi}_t\right)'$ for all $i > 0$.[14] We formalize this by specifying the subjective scenario as

$$\tilde{\Theta}_t = \left\{\tilde{\theta}_{t,t+1}, \tilde{\theta}_{t,t+2}, \ldots\right\} = \tilde{\Theta}^k = \left\{\tilde{\theta}^k, \tilde{\theta}^k, \ldots\right\}, \qquad \tilde{\theta}^k = \left(\tilde{\omega}^k, \tilde{\phi}^k\right)' \in I^\theta, \tag{15}$$

for $t = \tilde{\tau}_{k-1}, \tilde{\tau}_{k-1}+1, \ldots, \tilde{\tau}_k - 1$ and where $k = 1, 2, \ldots$ denotes the different subperiods with $\tilde{\tau}_k > \tilde{\tau}_{k-1}$ and $\tilde{\tau}_0 = 1$. We assume that either $\tilde{\omega}^k \neq \tilde{\omega}^{k-1}$, $\tilde{\phi}^k \neq \tilde{\phi}^k$, or both, such that at least one of the two subjective parameters changes from one subperiod to the next. This formalization represents the subjective parameters as being constant at all times during subperiod $k$, where $\tilde{\Theta}_t = \tilde{\Theta}^k$, and across all forecast horizons $i > 0$, but undergoing nonrecurring structural shifts at times $\{\tilde{\tau}_1, \tilde{\tau}_2, \ldots\}$. As with the objective KU parameters, the timing and magnitude of future shifts in these subjective parameters are unforeseeable *ex ante.*

Because market participants understand that the inflation process undergoes unforeseeable change only intermittently, we would expect their subjective assessment of the unknown future objective KU parameters to be related to the current values of these objective parameters. However, we would expect this relationship to be imprecise: the subjective parameters might shift in anticipation of future shifts in the objective parameters prior to these shifts, or after shifts in the objective parameters in response to these shifts. Moreover, the magnitudes of the shifts in the subjective and objective parameters need not fully match.

We formalize these ideas with an anchoring constraint: we assume that the absolute difference between the subjective parameters, $\tilde{\theta}_t$, and the current objective KU parameters, $\bar{\theta}_t$, is small relative to the model's interval for these parameters, $I^\theta$, which represents their largest possible difference. We formalize this by assuming that there exists a $0 < \delta < 1$, such that, at each time $t$,

$$|\tilde{\omega}_t - \bar{\omega}_t| \leq \delta\left(\omega_U - \omega_L\right), \qquad \left|\tilde{\phi}_t - \bar{\phi}_t\right| \leq \delta\left(\phi_U - \phi_L\right). \tag{16}$$

The difference $\tilde{\omega}_t - \bar{\omega}_t$ can be interpreted as participants' subjective assessment of the future change in the objective KU parameters. Thus, the anchoring constraint formalizes the bounds on participants' subjective assessment of future changes in the objective KU parameters.

With this specification of participants' subjective scenario, $\tilde{\Theta}_t$, and the unique solution for the scenario-consistent expectation in (11) in Proposition 1, participants'

[14] This assumption is needed neither to represent participants' expectations, as illustrated by Proposition 1, nor to derive the reduced-form expression for inflation. However, it simplifies these expressions, thereby allowing a direct comparison between KMH's and REH's representations of the inflation process.

inflation forecast in (2) can be expressed, during subperiod $k$, as

$$F_t\left(\pi_{t+1}\right) = E_t\left(\pi_{t+1}\left(\tilde{\Theta}^k\right)\right) = \tilde{\omega}^k + \frac{\kappa\tilde{\phi}^k}{1-\beta\tilde{\phi}^k}y_t, \tag{17}$$

for $t = \tilde{\tau}_{k-1}, \tilde{\tau}_{k-1}+1, \ldots, \tilde{\tau}_k - 1$ and $k = 1, 2, \ldots$. This represents participants' expectation of inflation as being consistent with the model's objective prediction of $\pi_{t+1}$, in the sense that it is the scenario-consistent expectation that corresponds to participants' subjective scenario.

Like REH, KMH's implementation of Muth's hypothesis represents participants' inflation expectation in terms of the relevant information set, according to the model. For the baseline NKPC model, this relevant information at time $t$ consists only of the output gap, $y_t$. Thus, KMH maintains Muth's two core assertions: information is not wasted and "the way expectations are formed depends specifically on the relevant system describing the economy" (Muth, 1961, p. 316).

However, by allowing for unforeseeable change, KMH, unlike REH, acknowledges that participants revise, at least intermittently, their forecasting strategy – how they map current information about the output gap, $y_t$, onto their expectation of future inflation, as determined by their subjective scenario's scenario – in anticipation of and response to change in the objective KU parameters. This implies that, beyond being driven by random shocks to the output gap, participants' inflation expectations are also driven over time by nonrecurring shifts in the objective parameter $\bar{\phi}_t$ and the subjective parameters $\tilde{\theta}_t$ on which the output gap and participants' expectations, respectively, depend. As the change in these parameters is unforeseeable, how participants' will revise their inflation expectations in the future is inherently unknown *ex ante*.

In contrast, because the constant-parameter REH model assumes $\bar{\theta}_t = \bar{\theta}$ for all $t$, such that $I^\theta = \left[\bar{\theta}, \bar{\theta}\right] = \bar{\theta}$ and $\tilde{\Theta}_t = \bar{\Theta} = \left\{\bar{\theta}, \bar{\theta}, \ldots\right\}$, it represents participants' expectation of inflation with the conditional expectation that constitutes the time-invariant model's only prediction of future inflation. As this is given by $E_t\left(\pi_{t+1}\left(\bar{\Theta}\right)\right) = \bar{\omega} + \left(\kappa\bar{\phi}/\left(1-\beta\bar{\phi}\right)\right)y_t$, it represents participants' expectations in terms of the constant objective parameters, $\bar{\theta}$, which implies that their expectations are time-invariant and driven solely by the random shocks to the output gap.

## 3.3 The Reduced-Form Expression for Inflation

Muth's hypothesis simultaneously determines the NKPC model's objective process for inflation and its representation of participants' expectation of inflation. First, the structure of the NKPC model's specification of inflation in (2)-(3) and the constraints on its objective KU parameters determine the structure and constraints on the model-consistent representation of participants' inflation expectation in (17). In turn, inserting the representation of participants' expectation of inflation into the specification of inflation in (2)-(3) enables deriving the model's objective process for inflation, as given by the model's reduced-form expression for inflation.

At a point in time $t$ when the objective and subjective parameters for some $j$ and $k$ are given by $\bar{\theta}_t = \bar{\theta}^j$ and $\tilde{\theta}_t = \tilde{\theta}^k$, this reduced-form expression becomes

$$\begin{aligned}
\pi_t &= \bar{\omega}^j + \beta\left(E_t\left(\pi_{t+1}\left(\tilde{\Theta}_t\right)\right) - \bar{\omega}^j\right) + \kappa y_t \\
&= \bar{\omega}^j + \beta\left(\left(\tilde{\omega}^k + \frac{\kappa\tilde{\phi}^k}{1-\beta\tilde{\phi}^k} y_t\right) - \bar{\omega}^j\right) + \kappa y_t \\
&= (1-\beta)\,\bar{\omega}^j + \beta\tilde{\omega}^k + \frac{\kappa}{1-\beta\tilde{\phi}^k} y_t. \qquad (18)
\end{aligned}$$

The reduced-form expression characterizes how, directly and through participants' expectation of inflation, the output gap drives inflation over time. Importantly, this relation depends both on the objective parameters, $\bar{\theta}^j$, and the subjective parameters, $\tilde{\theta}^k$. As both of these parameters undergo nonrecurring shifts, the relationship between inflation and the output gap changes in nonrecurring ways at times $\{\bar{\tau}_j\}_{j=1}^{\infty} \cup \{\tilde{\tau}_k\}_{k=1}^{\infty}$.

### 3.3.1 An Autonomous Role for Participants' Inflation Expectations

The reduced-form expression for inflation in (18) illustrates a crucial implication of KMH: participants' expectation of inflation plays an autonomous role in driving inflation over time. Although the baseline NKPC specification of inflation in (2)-(3) determines the set of scenario-consistent expectations of future inflation, it does not determine which of the conditional expectations in this set represents participants' expectations. This is because KMH constrains the subjective parameters only to lie within the model's interval for the objective KU parameters. Importantly, it does not specify precisely the relationship between the subjective parameters and the objective KU parameters. Consequently, KMH implies that participants' inflation expectations play an autonomous role in driving inflation over time: participants decide when and how they revise their subjective assessments of the future objective KU parameters, and thus their subjective expectation of inflation, autonomously of the model's specification of the objective KU parameters.

In constrast, REH's implementation of Muth's hypothesis in a time-invariant model does not accord participants' expectations an autonomous role in driving aggregate outcomes. Constraining the NKPC model's objective parameters to remain constant over time, $\bar{\theta}_t = \bar{\theta}$ for all $t$, implies that participants' subjective parameters are *completely* determined by these constant parameters, $\tilde{\Theta}_t = \bar{\Theta} = \left\{\bar{\theta}, \bar{\theta}, \ldots\right\}$ for all $t$. Thus, participants' expectation of inflation is fully determined by the model's specification of inflation, and the reduced-form expression for inflation in (18) becomes $\pi_t = \bar{\omega} + \left(\kappa / \left(1 - \beta\bar{\phi}\right)\right) y_t$. Because this depends only on the constant objective parameters, $\bar{\theta}$, participants' expectation of inflation does not play an autonomous role in driving inflation over time.

An important implication of participants' inflation expectations playing an autonomous role in the KMH model is that participants' forecast errors are correlated with the current output gap, $y_t$. This is because participants' inflation expectation at time $t$ is based on the subjective scenario, which, in general, differs from the future objective KU parameters and participants' subjective scenario in the future.

To demonstrate this point, consider the case where $\tilde{\Theta}_t = \tilde{\Theta}_{t+1} = \tilde{\Theta}^k$ and $\bar{\theta}_t = \bar{\theta}_{t+1} = \bar{\theta}^j$. This implies that the *ex post* forecast error at time $t+1$ becomes:

$$\begin{aligned} \pi_{t+1} - E_t\left(\pi_{t+1}\left(\tilde{\Theta}^k\right)\right) &= (1-\beta)\left(\bar{\omega}^j - \tilde{\omega}^k\right) + \left(\bar{\phi}^j - \tilde{\phi}^k\right)\left(\frac{\kappa}{1-\beta\tilde{\phi}^k}\right) y_t \\ &\quad + \frac{\kappa\bar{\phi}^j}{1-\beta\tilde{\phi}^k}\eta_{t+1}. \end{aligned} \tag{19}$$

This *ex post* forecast error is correlated with the current output gap, $y_t$, and has a non-zero level determined by $\left(\bar{\omega}^j - \tilde{\omega}^k\right)$. The only exception is where $\bar{\omega}^j = \tilde{\omega}^k$ and $\bar{\phi}^j = \tilde{\phi}^k$, in which case participants' inflation expectation at time $t$ is *ex ante* optimal as it is based exactly on the subjective and objective KU parameters that characterize inflation at time $t+1$. However, because the forecast errors arise from unforeseeable change in both the objective KU parameters and the parameters of participants' subjective scenario, they are unknown *ex ante*. In contrast to time-invariant models, this implies that the information embedded in *ex post* forecast errors cannot be used to form inflation expectations that are *ex ante* optimal.

## 3.4 An Autoregressive Process for Inflation and Inflation Expectations

The reduced-form expression for inflation in (18) implies that inflation can be characterized with an autoregressive process with structural shifts in its parameters, as given by

$$\pi_t = \alpha^{j,k} + \rho^j \pi_{t-1} + \varepsilon_t, \tag{20}$$

where

$$\alpha^{j,k} = \left(1-\rho^j\right)\left((1-\beta)\,\bar{\omega}^j + \beta\tilde{\omega}^k\right), \qquad \rho^j = \bar{\phi}^j,$$

and $\varepsilon_t = \phi^j \eta_t$. This corresponds to the autoregressive process for inflation with structural shifts in (1). In Section 2, we presented empirical findings that the parameters of such a process do indeed undergo nonrecurring structural shifts, and that once these shifts are accounted for, the process is an adequate representation of the inflation time-series data.

Equivalently, KMH also implies that the representation of participants' expectation of inflation in (17) can be characterized with an autoregressive process with structural shifts in its parameters, as given by

$$F_t\left(\pi_{t+1}\right) = E_t\left(\pi_{t+1}\left(\tilde{\Theta}^k\right)\right) = \tilde{\alpha}^k + \rho^j E_{t-1}\left(\pi_t; \tilde{\Theta}^k\right) + \tilde{\varepsilon}_t, \tag{21}$$

where

$$\tilde{\alpha}^k = \left(1 - \rho^j\right) \tilde{\omega}^k, \qquad \rho^j = \bar{\phi}^j,$$

and $\tilde{\varepsilon}_t = \frac{\kappa \tilde{\phi}^k}{1-\beta\tilde{\phi}^k} \eta_t$.

The model implies that the persistence parameters in the autoregressive models for inflation and participants' expectation of inflation in (20) and (21) are identical and determined by the persistence of the output gap, $\bar{\phi}^j$. Moreover, the anchoring constraint in (16) restricts the difference between the levels that inflation and inflation expectations fluctuate around:

$$\left| \frac{\tilde{\alpha}^j}{1-\rho^j} - \frac{\alpha^{j,k}}{1-\rho^j} \right| = (1-\beta) \left| \tilde{\omega}^k - \bar{\omega}^j \right| \leq \delta \left( \omega_U - \omega_L \right).$$

In the next Section, we estimate a first-order autoregressive process for survey data on participants' inflation expectations, and we assess KMH's prediction that the parameters of this process undergo nonrecurring structural shifts.

# 4 Evidence of Structural Shifts in an Autoregressive Specification for Survey Forecasts of Inflation

A novel implication of imposing KMH in our baseline specification of the NKPC model is that both inflation and inflation expectations can be characterized by a first-order autoregressive process whose parameters undergo shifts. We presented our findings of such structural shifts in the first-order autoregressive process for inflation in Section 2. Moreover, once the structural shifts were accounted for, the estimated model was found to be an adequate representation of the inflation time-series data.

Here, we use survey forecasts of inflation from the Survey of Professional Forecasters to test empirically whether the parameters of a first-autoregressive process for the survey forecasts undergo structural shifts and whether such a process is an adequate representation of the data. To this end, we consider the quarterly data on the one-quarter-ahead PCE inflation, which we denote by $F_t^s(\pi_{t+1})$, covering the sample period of quarterly observations from 1968:Q4 to 2022:Q1. Figure 4 shows the actual inflation data and the one-quarter-ahead survey inflation forecasts. A full description of the data is given in Online Appendix B.1. The two series follow each other fairly closely over the sample period, but with subperiods during which next quarter's inflation is forecasted to lie above or below its current level.

We estimate the model given by

$$F_t^s(\pi_{t+1}) = \tilde{\rho}^j F_{t-1}^s(\pi_t) + \tilde{\mu}^j + \tilde{\varepsilon}_t, \tag{22}$$

for $t = \tilde{T}_{j-1}, \tilde{T}_{j-1}+1, \ldots, \tilde{T}_j - 1$ and $j = 1, 2, \ldots, \tilde{K}$, where $\tilde{T}_0 = 1$, $\tilde{T}_{\tilde{K}} = T + 1$, either $\tilde{\rho}^j \neq \tilde{\rho}^{j-1}$, $\tilde{\mu}^j \neq \tilde{\mu}^{j-1}$, or both, $0 \leq \tilde{\rho}^j < 1$ for all $j$, $\tilde{\varepsilon}_t \sim iidN\left(0, \tilde{\sigma}^2\right)$, and the

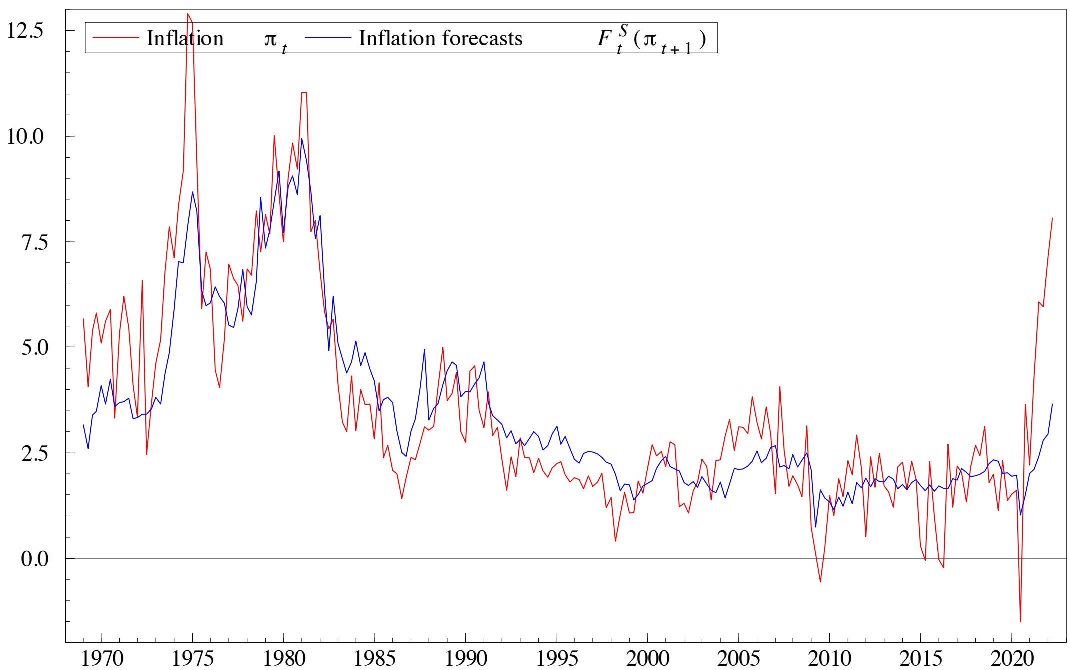


Figure 4: The figure shows the actual inflation $\pi_t$ (red line) and the one-quarter ahead survey inflation forecast $F_t^s(\pi_{t+1})$ (blue line).

initial value $F_0^S(\pi_1)$ is given. We use an effective estimation sample of 213 observations covering the period from 1969:Q1 to 2022:Q1. We estimate the process both with constant parameters, $\left(\tilde{\rho}^l, \tilde{\mu}^l\right) = (\tilde{\rho}, \tilde{\mu})$ for all $t = 1, 2, \ldots, T$, and with structural shifts in the parameters $\left(\tilde{\rho}^l, \tilde{\mu}^l\right)$. As in Section 2, we use the Autometrics algorithm with step-indicator and multiple step-indicator saturation to identify the structural shifts, as outlined in Online Appendix B.2.[15]

The empirical estimates for the autoregressive process in (21) with constant parameters are shown in the left columns of Table 2 and the fit of the model, including rolling window estimates of the parameters, is illustrated in Figure 5. With constant parameters, the estimated persistence is 0.968 (std. error of 0.017) which means that the inflation forecast is essentially a random walk with a drift. However, the rolling window estimates of the parameters, shown in panels (c) and (d) in Figure 5, suggest that $(\tilde{\rho}, \tilde{\mu})$ are not constant over time. For example, the rolling window estimates of $\tilde{\rho}$ fluctuate a lot in the beginning of the sample, though they seem more stable during the latter part of the sample period.

The estimates with structural shifts in the parameters $\left(\tilde{\rho}^j, \tilde{\mu}^j\right)$ are shown in the right

[15]For a presentation of Autometrics and an analysis of its properties, see Doornik (2009), Castle *et al.* (2012), and Castle *et al.* (2015).

Table 2: Estimates of the Autoregressive Process for Inflation Forecasts with Constant Parameters and with Structural Shifts in the Parameters

| Model with constant parameters | | | Model with structural shifts | | | |
|---|---|---|---|---|---|---|
| Parameter | Estimate | Std. error | Parameter, period | | Estimate | Std. error |
| $\tilde{\rho}$ | 0.968 | 0.017 | $\tilde{\rho}$ | 69:1-73:1 | 0.256 | 0.097 |
| | | | | 73:2-79:3 | 0.478 | 0.072 |
| | | | | 79:4-80:3 | 0.321 | 0.072 |
| | | | | 80:4-80:4 | 0.448 | 0.083 |
| | | | | 81:1-81:4 | 1.372 | 0.083 |
| | | | | 82:1-82:3 | 1.093 | 0.070 |
| | | | | 82:4-86:3 | 0.769 | 0.045 |
| | | | | 86:4-87:3 | 0.966 | 0.074 |
| | | | | 87:4-87:4 | 0.509 | 0.074 |
| | | | | 88:1-22:1 | 0.833 | 0.043 |
| $\tilde{\mu}$ | 0.110 | 0.068 | $\tilde{\mu}$ | 69:1-73:3 | 2.667 | 0.340 |
| | | | | 73:4-75:1 | 4.158 | 0.511 |
| | | | | 75:2-78:2 | 3.116 | 0.454 |
| | | | | 78:3-78:3 | 5.411 | 0.566 |
| | | | | 78:4-78:4 | 3.265 | 0.689 |
| | | | | 79:1-79:4 | 4.730 | 0.590 |
| | | | | 80:1-80:4 | 6.084 | 0.642 |
| | | | | 81:1-81:3 | -4.260 | 0.796 |
| | | | | 81:4-82:2 | -2.275 | 0.552 |
| | | | | 82:3-90:4 | 0.767 | 0.184 |
| | | | | 91:1-22:1 | 0.346 | 0.095 |
| $\sigma$ | | 0.506 | | | | 0.316 |
| $R^2$ | | 0.94 | | | | 0.98 |
| Observations | | 213 | | | | 213 |
| Misspecification tests | | [p-value] | | | | [p-value] |
| No autocorr., order 1-2 | | [0.405] | | | | [0.784] |
| No ARCH, order 1-4 | | [0.000] | | | | [0.558] |
| No heteroskedasticity | | [0.000] | | | | [0.905] |
| Normality | | [0.000] | | | | [0.000[ |

Notes: The table shows estimates of the autoregressive process for the survey inflation forecasts in (22) with constant parameters and with structural shifts in the parameters. The effective estimation sample is quarterly observations from 1969:Q1 to 2022Q:1.

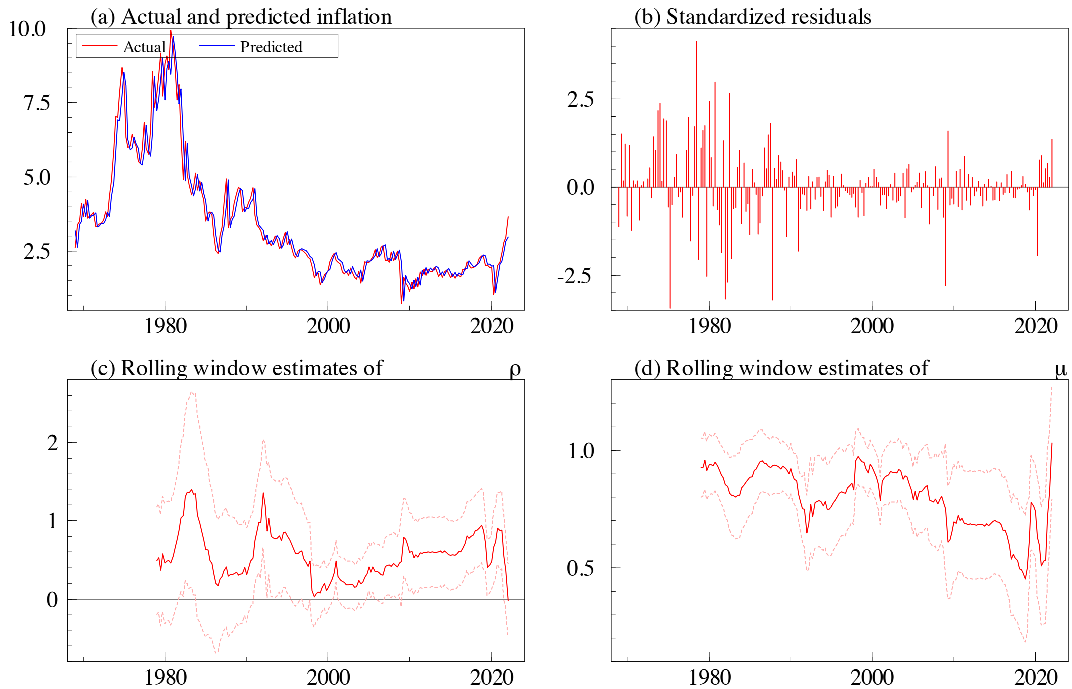


Figure 5: The plot illustrates the estimated autoregressive process for the survey inflation forecasts $F_t^s(\pi_{t+1})$ in (22) with constant parameters. Panel (a) shows the actual inflation forecast $F_t^s(\pi_{t+1})$ (red line) and the model's predicted inflation forecasts (blue line). Panel (b) shows the standardized estimated residuals. Panels (c) and (d) show the rolling window estimates of the parameters $\tilde{\rho}$ and $\tilde{\mu}$ (solid red lines) and their 95 percent confidence intervals (dotted red lines) based on a 10-year rolling window sample that ends at the point in time illustrated by the red lines.

columns of Table 2 and the fit of the model, including illustrations of the estimates of $\left(\tilde{\rho}^j, \tilde{\mu}^j\right)$ over time, is illustrated in Figure 6. We find evidence of structural shifts in both parameters. The majority of these happened in the 1970s and the early 1980s before the Great Moderation, after which the parameters remain stable. In the 1970s and until 1981, we the estimated persistence $\tilde{\rho}^j$ ranges from 0.256 (std. error of 0.097) to 0.448 (std. error of 0.083). This low persistence is combined with high estimates of $\tilde{\mu}^j$ ranging from 2.667 (std. error of 0.340) to 5.411 (std. error of 0.566), which implies that the inflation forecasts fluctuate around a level $\frac{\tilde{\mu}^j}{1-\tilde{\rho}^j}$ (given that $0 \leq \tilde{\rho}^j < 1$) ranging from 3.6 at the beginning of the 1970s to higher and rapidly changing levels from 1973.

From 1982/1983, the estimated persistence increases and settles at 0.833 (std. error of 0.043) in 1988:Q1 and remains stable for the rest of the sample period considered. This increasing persistence is combined with a lower estimates of $\tilde{\mu}^j$ that drop from 0.767 (std. error of 0.184) over the subperiod from 1982:Q3-1990:Q4 to 0.346 (std.

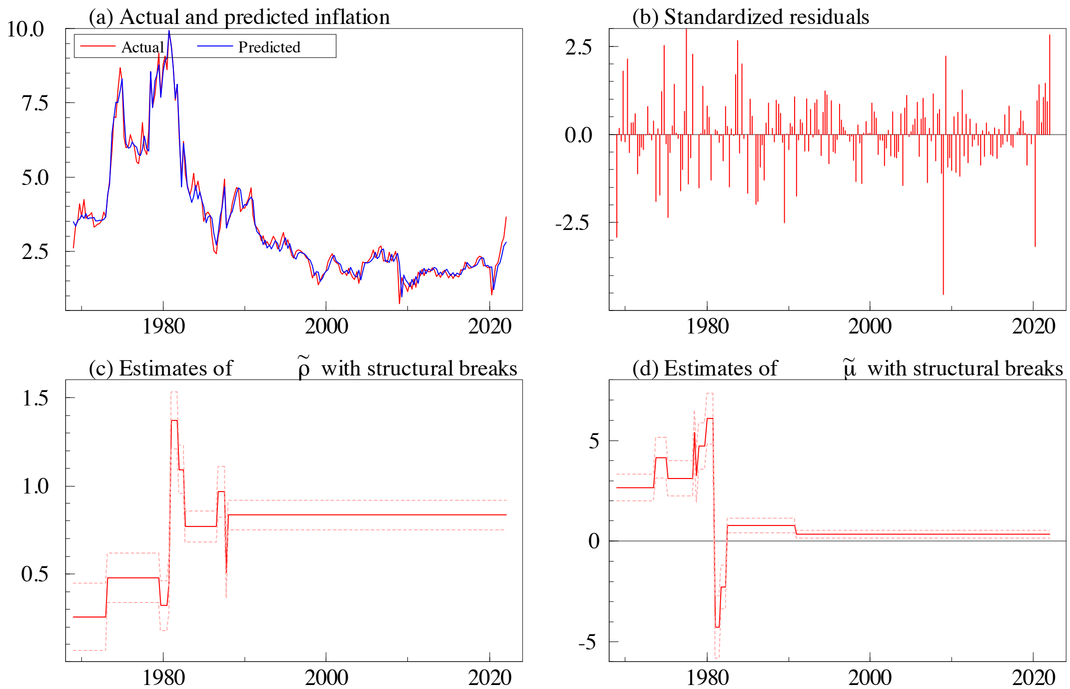


Figure 6: The plot illustrates the estimated autoregressive process for the survey inflation forecasts $F_t^s(\pi_{t+1})$ in (22) with structural shifts in the parameters. Panel (a) shows the actual inflation forecasts $F_t^s(\pi_{t+1})$ (red line) and the model's predicted inflation forecasts (blue line). Panel (b) shows the standardized estimated residuals. Panels (c) and (d) show the estimates of $\tilde{\rho}^j$ and $\tilde{\mu}^j$ with structural shifts (solid red lines) and their 95 percent confidence intervals (dotted red lines).

error of 0.095) for the rest of the sample period considered. As both $\tilde{\rho}^j$ and $\tilde{\mu}^j$ stabilize, the inflation forecasts fluctuate around a constant level of 2.1 from 1991:Q1 and until the end of the sample period.

The misspecification tests reveal that the autoregressive process becomes a better approximation of the inflation forecast data when structural shifts are accounted for: the tests do not reject the null of no autocorrelation, no ARCH, and no heteroskedasticity of the residuals with high p-values. However, the null of normality of the residuals is rejected.

The baseline NKPC model under KMH implies that the persistence parameter in the autoregressive specifications for both inflation and inflation forecasts should be the same during each subperiod, and that it is given by the current persistence parameter in the autoregressive process for the output gap, $\phi^j$. Thus, according to the theory, $\rho^j = \tilde{\rho}^j = \phi^j$. Panel (a) in Figure 7 shows that the estimates of $\rho^j$ and $\tilde{\rho}^j$ follow each other over the sample period and that most of the structural breaks in both parameters occured before the Great Moderation. However, during the Great Moderation period,

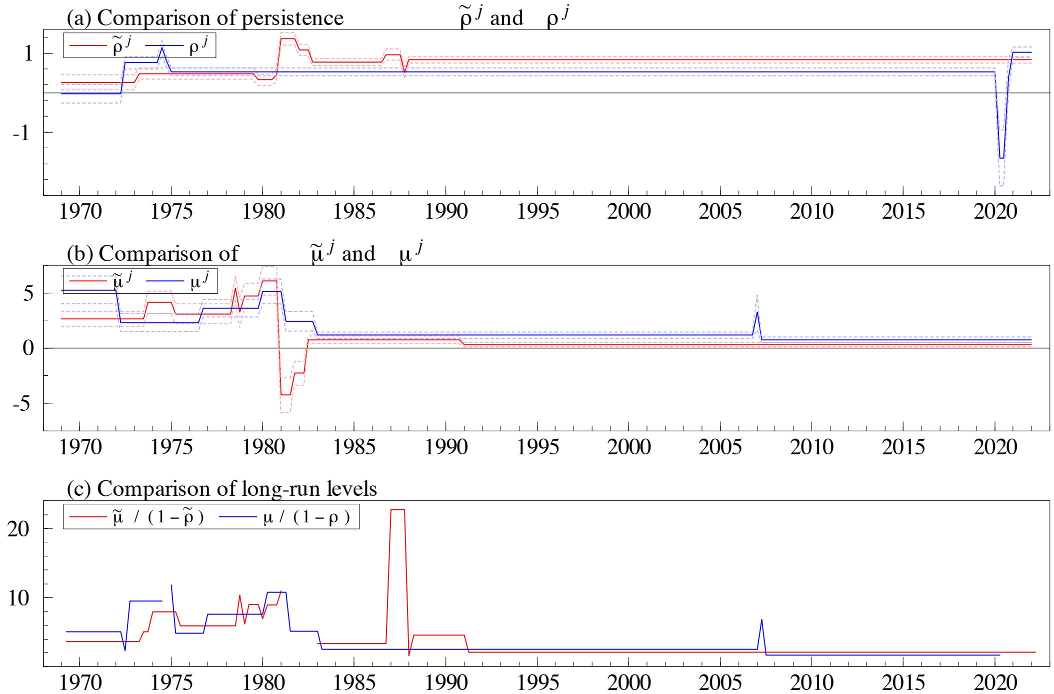


Figure 7: The figure compares the estimates of $\left(\tilde{\rho}^j, \tilde{\mu}^j\right)$ from the autoregressive process for the survey inflation forecasts in (22) and $(\rho^j, \mu^j)$ from the autoregressive process for inflation in (1). Panel (a) shows the estimates of $\tilde{\rho}^j$ (red line) and $\rho^j$ (blue line) with their 95 percent confidence intervals (dotted red and blue lines). Similarly, panel (b) shows the estimates of $\tilde{\mu}^j$ (red line) and $\mu^j$ (blue lines) and their 95 percent confidence intervals (dotted red and blue lines). Panel (c) shows the estimates $\tilde{\mu}^j / \left(1 - \tilde{\rho}^j\right)$ and $\mu^j / (1 - \rho^j)$, but excluding the observations where $\tilde{\rho}^j$ and $\rho^j$ are either smaller than $-1$ or larger than 1.

the estimated $\tilde{\rho}^j$ is slightly greater than $\rho^j$, and the difference between the two is statistically significantly.

The greater persistence in the inflation forecasts during the Great Moderation, however, is offset by a lower estimate of $\tilde{\mu}^j$ relative to $\mu^j$ during that period. As illustrated in Panel (b) in Figure 7, during the 1970 and the early 1980s, both $\tilde{\mu}^j$ and $\mu^j$ undergo frequent and large structural shifts. Moreover, while the shifts in $\tilde{\mu}^j$ sometimes exceed those in $\mu^j$, several times $\tilde{\mu}^j$ shifts less than $\mu^j$. However, as both inflation forecasts and inflation start to stabilize from 1982, both $\tilde{\mu}^j$ and $\mu^j$ stabilize with the former estimated to be lower than the latter during the stable period during the Great Moderation. Because $\tilde{\rho}^j > \rho^j$ and $\tilde{\mu}^j < \mu^j$ during the Great Moderation, both inflation forecasts and inflation fluctuate around levels, $\tilde{\mu}^j / \left(1 - \tilde{\rho}^j\right)$ and $\mu^j / (1 - \rho^j)$, that are almost identical during this period, as illustrated by Panel (c) in Figure 7.

These empirical findings are broadly consistent with the implication of the baseline NKPC model under KMH. We find evidence of structural breaks in an autoregressive process for the survey inflation forecasts and the estimated model is found to be an adequate representation of the survey data. However, the combination of slightly higher estimated persistence of inflation forecasts and slightly lower $\tilde{\mu}^j$, relative to $\rho^j$ and $\mu^j$, during the Great Moderation period might suggest that the inflation forecasts are, at least partly, driven by some factors not captured by the simple baseline NKPC model that we used to illustrate KMH. We leave an exploration of these findings to future research.

# 5 Concluding Remarks

Recognizing ambiguity about the process driving outcomes is increasingly viewed as crucial to remedying the shortcomings of REH models. For example, Hansen (2013) argues that REH models "miss something *essential*: uncertainty [arising from] ambiguity about which is the correct model" of aggregate outcomes (p. 399, emphasis added).

This paper contributes to the rapidly growing literature recognizing uncertainty arising from model ambiguity in macroeconomics and finance theory. In an authoritative review, Illut and Schneider (2022, p. 30) identified an important gap in this literature:

> Our review... [indicates] that the literature has made a lot of progress understanding how beliefs shape data. We feel that for the other direction, how data shape beliefs, existing quantitative work has only scratched the surface, in particular with respect to learning about a world that is *constantly evolving* due to structural change (emphasis added).

The Knight-Muth hypothesis makes three contributions to filling this gap. First, by opening a macroeconomic model to nonrecurring shifts in its parameters, KMH provides a tractable and empirically testable formalization of model ambiguity in "a world that is constantly evolving due to structural change." Second, KMH's interval and anchoring constraints formalize participants' "learning" – how they revise their forecasting strategies – when faced with nonrecurring structural shifts. Lastly, by imposing model-consistency under Knightian uncertainty, KMH represents a two-way interdependence between "how beliefs shape data [and] how data shape beliefs."

Contextualizing KMH within the varied model-ambiguity literature is beyond the scope of this paper. Here, we offer a few remarks comparing KMH's model-consistent representations of model ambiguity arising from unforeseeable change with a substantial part of the literature that formalizes ambiguity in time-invariant models.

Contemporary models that have acknowledged the relevance of model ambiguity have typically focused on participants' ambiguity about the process driving outcomes and represented it with a *set* of time-invariant processes. For example, Hansen and Sargent (2008) developed an influential approach to building models recognizing participants' ambiguity. Aiming to "preserve much of the discipline of rational expectations," they "impose a common approximating [time-invariant] model on all decision makers" However, they allow [participants] to express different degrees of mistrust in that model" and assume that this mistrust engenders ambiguity about the process driving outcomes, which they formalize with "the set of models (...) surrounding the approximating model" (pp. 7, 11).

Hansen and Sargent (p. 17) acknowledge that, as the sample of observations increases, standard estimation procedures would enable participants to "learn the correct [time-invariant] specification" with increasing accuracy, thereby reducing the extent of their ambiguity. Indeed, in time-invariant models, participants' ambiguity disappears asymptotically. Thus, in Epstein and Schneider's (2007) nomenclature, it is "resolvable."

To illustrate the difference between resolvable and unresolvable model ambiguity, Epstein and Schneider consider an individual with intertemporal multiple-priors utility who initially faces both resolvable and unresolvable ambiguity.[16] The resolvable ambiguity can be formalized, for example, as the ambiguity about an unknown parameter in a time-invariant process for outcomes. In contrast, the unresolvable ambiguity arises from nonrepetitive change that resembles what we call unforeseeable change. Epstein and Schneider show that with an appropriate learning mechanism, the resolvable ambiguity converges towards the objective process asymptotically; ultimately, only the unresolvable ambiguity remains.

This paper contributes to the model-ambiguity literature by implementing Muth's hypothesis in a model that recognizes that an economist and market participants face unresolvable ambiguity about the process driving outcomes, which arises from unforeseeable change in the parameters of that process. Applying KMH in a baseline New Keynesian Phillips curve model for inflation represents the model's prediction of future inflation with a set of scenario-consistent expectations and processes. Importantly, because it is unknown *ex ante* when and how the future parameters will shift, this set formalizes unresolvable ambiguity about the inflation process on the part of an economist and market participants. KMH can be combined with multiple-priors utility by specifying ambiguity averse participants' set of expectations with a subset of the model-implied scenario-consistent expectations. However, KMH implies that the endpoints of this subset must be open to unforeseeable change.

Our findings in this paper suggest that acknowledging that ambiguity about the inflation process is unresolvable and implementing Muth's hypothesis under such ambiguity are crucial for understanding how inflation and participants' expectation of inflation, as measured by survey forecasts, unfold over time.

[16] According to the multiple-priors utility approach, an ambiguity-averse individual's optimal decision maximizes the minimum expected utility over some set of subjective expectations. Gilboa and Schmeidler (1989) axiomatized multiple-priors utility in a static setting and Epstein and Schneider (2003) extended their axiomatization to an intertemporal setting.

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# A Appendix

## A.1 Additional Empirical Estimates for the Autoregressive Process for Inflation

In this section, we present empirical estimates of the autoregressive process for inflation with structural shifts in (1) extended to include four lags of inflation. Specifically, we estimate the process given by

$$\pi_t = \rho_1^j \pi_{t-1} + \rho_2 \pi_{t-2} + \rho_3 \pi_{t-3} + \rho_4 \pi_{t-t} + \mu^j + \varepsilon_t, \tag{23}$$

for $t = T_{j-1}, T_{j-1} + 1, \ldots, T_j - 1$ and $j = 1, 2, \ldots, K$, where $T_0 = 1$, $T_K = T + 1$, $\varepsilon_t \sim iidN(0, \sigma^2)$, and the initial values $(\pi_0, \pi_{-1}, \pi_{-2}, \pi_{t-3})$ are given.

The lagged inflation variables are included such that they are always retained Autometrics's selection procedure.

Table 3 presents the empirical results with and without structural breaks in the parameters of the augmented model in (23). The parameters $\rho_2$ and $\rho_4$ are estimated as insignificantly and borderline insignificantly different from zero. Although augmenting the autoregressive process with lags of inflation changes the timing and magnitude of the shifts in the parameters $\left(\rho_1^j, \mu^j\right)$ marginally, it does not change the conclusions of the empirical results presented in Section 2.

## A.2 Proofs

**Proof of Proposition 1.** By forward-iteration with respect to the process for future inflation and output gap $\{\pi_{t+i}(\Theta_t), y_{t+i}(\Theta_t)\}_{i=1}^{\infty}$ in (7)-(8) and using the law of iterated expectations, the scenario-consistent expectation $E_t(\pi_{t+1}(\Theta_t))$ can be expressed as:

$$\begin{aligned}
E_t(\pi_{t+1}(\Theta_t)) &= E_t(\omega_{t,t+1} + \beta(E_{t+1}(\pi_{t+2}(\Theta_t)) - \omega_{t+1}) + \kappa y_{t+1}(\Theta_t)) \\
&= (1-\beta)\,\omega_{t,t+1} + \beta E_t(\pi_{t+2}(\Theta_t)) + \kappa E_t(y_{t+1}(\Theta_t)) \\
&= (1-\beta)\,\omega_{t,t+1} + \beta E_t(\omega_{t,t+2} + \beta(E_{t+2}(\pi_{t+3}(\Theta_t)) - \omega_{t,t+2}) + \kappa y_{t+2}(\Theta_t)) \\
&\qquad + \kappa E_t(y_{t+1}(\Theta_t)) \\
&= (1-\beta)\,\omega_{t,t+1} + (1-\beta)\,\beta\omega_{t,t+2} + \beta^2 E_t(\pi_{t+3}(\Theta_t)) + \kappa E_t(y_{t+1}(\Theta_t)) \\
&\qquad + \kappa\beta E_t(y_{t+2}(\Theta_t)).
\end{aligned}$$

Table 3: Estimates of the Autoregressive Process for Inflation with Constant Parameters and with Structural Shifts in the Parameters Augmented With Lags of Inflation

| Model with constant parameters | | | Model with structural shifts | | | |
|---|---|---|---|---|---|---|
| Parameter | Estimate | Std. error | Parameter, period | | Estimate | Std. error |
| $\rho_1$ | 0.656 | 0.069 | $\rho_1^j$ | 69:1-74:4 | 0.766 | 0.068 |
| | | | | 75:1-20:1 | 0.418 | 0.063 |
| | | | | 20:2-20:3 | -1.677 | 0.356 |
| | | | | 20:4-22:1 | 0.376 | 0.154 |
| $\rho_2$ | 0.179 | 0.082 | $\rho_2$ | | 0.082 | 0.062 |
| $\rho_3$ | 0.118 | 0.082 | $\rho_3$ | | 0.147 | 0.059 |
| $\rho_4$ | -0.023 | 0.070 | $\rho_4$ | | 0.102 | 0.051 |
| $\mu$ | 0.259 | 0.138 | $\mu^j$ | 69:1-72:1 | 2.413 | 0.745 |
| | | | | 72:2-72:2 | -4.023 | 0.893 |
| | | | | 72:3-74:2 | 0.514 | 0.478 |
| | | | | 74:3-74:3 | 3.319 | 0.997 |
| | | | | 74:4-76:2 | -0.062 | 0.453 |
| | | | | 76:3-81:1 | 2.435 | 0.333 |
| | | | | 81:2-20:4 | 0.520 | 0.112 |
| | | | | 21:1-22:1 | 3.366 | 0.887 |
| $\sigma$ | | 1.125 | | | | 0.780 |
| $R^2$ | | 0.81 | | | | 0.91 |
| Observations | | 213 | | | | 213 |
| Misspecification tests | | [p-value] | | | | [p-value] |
| No autocorr., order 1-2 | | [0.797] | | | | [0.289] |
| No ARCH, order 1-4 | | [0.000] | | | | [0.115] |
| No heteroskedasticity | | [0.000] | | | | [0.075] |
| Normality | | [0.000] | | | | [0.832] |

Notes: The table shows estimates of the autoregressive process for inflation augmented with lags of inflation in (23). The columns to the left show the estimates of the model with constant parameters and the columns to the right shows the estimates with structural shifts in the parameters. The estimation sample is quarterly observations from 1969:Q1 to 2022Q:1.

Continuing this iteration yields:

$$
\begin{aligned}
E_t\left(\pi_{t+1}\left(\Theta_t\right)\right) &= (1-\beta)\sum_{i=1}^{n}\beta^{i-1}\omega_{t,t+i} + \beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right) + \kappa\sum_{i=1}^{n}\beta^{i-1}E_t\left(y_{t+i}\left(\Theta_t\right)\right) \\
&= (1-\beta)\sum_{i=1}^{n}\beta^{i-1}\omega_{t,t+i} + \beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right) \\
&\qquad +\kappa\sum_{i=1}^{n}\beta^{i-1}\left(\prod_{j=1}^{i}\phi_{t,t+j}\right)y_t. \qquad (24)
\end{aligned}
$$

We next show that the first and last terms in (24) are convergent for $n \to \infty$, which implies that $\lim_{n\to\infty}\beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right)$ exists.

Define $S_n = \sum_{i=1}^{n-1}\beta^{i-1}\left(\prod_{j=1}^{i}\phi_{t,t+j}\right)$. Because $0 \leq \phi_L \leq \phi_{t,t+i} \leq \phi_U < 1$ for all $i > 0$ and $0 < \beta < 1$, it follows that

$$
S_n - S_{n-1} = \beta^{n-2}\prod_{j=1}^{n-1}\phi_{t,t+j} > 0, \qquad \text{and} \qquad 0 \leq S_n \leq \sum_{i=1}^{n-1}\beta^{i-1} \leq \frac{1}{1-\beta}.
$$

This shows that $S_n$ is a bounded increasing sequence and hence is convergent for $n \to \infty$.

Equivalently, define $Z_n = \sum_{i=1}^{n-1}\beta^{i-1}\omega_{t,t+i}$. Because $0 < \omega_L \leq \omega_{t,t+i} \leq \omega_U$ for all $i$ and $0 < \beta < 1$, it follows that

$$
Z_n - Z_{n-1} = \beta^{n-2}\omega_{t,t+n-1} > 0, \qquad \text{and} \qquad 0 \leq Z_n \leq \sum_{i=1}^{\infty}\beta^{i-1}\omega_U = \frac{\omega_U}{1-\beta}.
$$

This shows that $Z_n$ is a bounded increasing sequence and hence is convergent for $n \to \infty$.

Because the first and last terms of (24) are convergent for $n \to \infty$, the term $\beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right)$ must also be convergent for $n \to \infty$.

Moreover, assuming that $\beta^n E_t\left(\pi_{t+n+1}\left(\Theta_t\right)\right) = 0$, as stated in (10), the scenario-consistent expectation $E_t\left(\pi_{t+1}\left(\Theta_t\right)\right)$ in (24) has the unique solution for $n \to \infty$:

$$
E_t\left(\pi_{t+1}\left(\Theta_t\right)\right) = (1-\beta)\sum_{i=1}^{\infty}\beta^{i-1}\omega_{t,t+i} + \kappa\sum_{i=1}^{\infty}\beta^{i-1}\left(\prod_{j=1}^{i}\phi_{t,t+j}\right)y_t. \qquad (25)
$$

■

# B Online Appendix

## B.1 Data

We estimate the autoregressive process for inflation in Section 2 using quarterly measures of the year-to-year change in the price index of the gross domestic product (PDGP) prepared by the Bureau of Economic Research. The data set has been downloaded from the Federal Reserve Bank of Philadelphia's website with data from the Survey of Professional Forecasters. We use the inflation variable `Most_Recent` from the data set `p_first_second_third.xlsx`, available at:
`https://www.philadelphiafed.org/surveys-and-data/real-time-data-research/p`.
This data set contains the initial vintage releases and the latest release of PGDP inflation. We have used the latest release of the data updated on May 26, 2022. The data are in percentage points and seasonally adjusted.

At the time we downloaded the data, the data set contained 227 quarterly observations covering the period from 1965:Q3 to 2022:Q1. However, the empirical results presented in Section 2 are based on an effective estimation sample from 1969:Q1 to 2022:Q1 to match the period for which survey inflation forecasts are available.

We estimate the autoregressive process for aggregate inflation forecasts in Section 4 using quarterly measured survey forecasts of the annualized one-quarter-ahead growth in the price level of the gross domestic product (PGDP). The data has been downloaded from the Federal Reserve Bank of Philadelphia's website with data from the Survey of Professional Forecasters. We use the mean forecast of the annualized one-quarter ahead PGDP inflation given by the variable `DPGDP3` in the data set `Mean_PGDP_Growth.xlsx` available at: `https://www.philadelphiafed.org/surveys-and-data/pgdp`.

The forecast data is available from 1968:Q4. We use an effective estimation sample from 1969:Q1 to 2022:Q1.

## B.2 Identifying Structural Shifts Using Autometrics with Step-Indicator and Multiplicative Step-Indicator Saturation

This section illustrates how we use the Autometrics algorithm with step-indicator and multiple step-indicator saturation to identify and estimate the structural shifts in the parameters of the autoregressive processes for inflation and survey inflation forecasts in Sections 2 and 4, respectively. We illustrate the approach for the autoregressive process for inflation, but an identical approach is used for the survey inflation forecast data.

To identify and estimate the structural shifts in the parameters $(\rho^j, \mu^j)$ in the autoregressive process in (1), we augment the model with step-indicator and multiplicative step-indicator variables (see Castle *et al.*, 2015). This transforms the identification and estimation of structural breaks into a model-selection problem that can be handled by the Autometrics tree-search algorithm (Doornik, 2009) in OxMetrics.

Specifically, the Autometrics model-selection algorithm starts from the general unrestricted model (GUM):

$$\pi_t = \rho\pi_{t-1} + \mu + \sum_{i=2}^{T} \Delta\rho_i \mathbf{1}(t \geq i)\pi_{t-1} + \sum_{i=2}^{T} \Delta\mu_i \mathbf{1}(t \geq i) + \varepsilon_t, \tag{26}$$

for $t = 1, 2, \ldots, T$ and where $\mathbf{1}(t \geq i)$ denotes an indicator variable that takes the value 1 for all $t \geq i$, and 0 otherwise; $\mathbf{1}(t = i)$ denotes dummy variables that take the value 1 at $t = i$, and 0 otherwise; $\rho$, $\mu$, $\{\Delta\rho_i, \Delta\mu_i\}_{i=2}^{T}$, and $\{\delta_i\}_{i=1}^{T}$ are parameters; and the initial value $\pi_0$ is given. In practice, the saturation with step-indicators and multiple step-indicators is done by creating the saturated variables $s_t^i = \mathbf{1}(t = i)$ and $\pi_t^i = \mathbf{1}(t \geq i)\pi_{t-1}$ for $i = 2, 3, \ldots, T$, such that the GUM can be identically written in terms of observable variables as:

$$\pi_t = \rho\pi_{t-1} + \mu + \sum_{i=2}^{T} \Delta\rho_i \pi_{t-1}^i + \sum_{i=2}^{T} \Delta\mu_i s_t^i + \varepsilon_t. \tag{27}$$

The GUM in (27) cannot be directly estimated as the number of candidate variables exceeds the number of observations. However, the Autometrics algorithm can handle model-selection in such situations. The algorithm delivers a selected terminal model that contains only the variables and parameters retained after model-selection. We use the Autometrics algorithm with a target size of 0.005.

A crucial advantage of this procedure is that it allows us to identify structural breaks in $\rho^j$ and $\mu^j$ at any time during the sample period by including the step-indicator variables $\mathbf{1}(t \geq i)$ and the multiplicative step-indicator variables $\mathbf{1}(t \geq i)\pi_{t-1}$ for $i = 2, 3, \ldots, T$. Importantly, the number and timing of breaks in $\rho$ and $\mu$ need not correspond. To illustrate, if the variable $\pi_{t-1}^{10}$ is retained by Autometrics in the selected terminal model, it corresponds to a structural break in the parameter $\rho^j$ at time $t = 10$ and with magnitude given by the parameter $\Delta\rho_{10}$.

The selected terminal model with $m$ structural breaks in $\rho^j$ at times $\{T_j^\rho\}_{j=1}^{m}$ and $n$ structural breaks in $\mu^j$ at times $\{T_j^\mu\}_{j=1}^{n}$ can be written as:

$$\pi_t = \rho\pi_{t-1} + \mu + \sum_{j=1}^{m} \Delta\rho_j \pi_{t-1}^j + \sum_{j=1}^{n} \Delta\mu_j s_t^j + \varepsilon_t. \tag{28}$$

For ease of interpretation, we present the estimation results of the selected terminal model in (28) rewritten in its equivalent representation:

$$\pi_t = \sum_{j=1}^{m+1} \rho^j \mathbf{1}\left(T_{j-1}^\rho \leq t < T_j^\rho\right)\pi_{t-1} + \sum_{j=1}^{n+1} \mu^j \mathbf{1}\left(T_{j-1}^\mu \leq t < T_j^\mu\right)\pi_{t-1} + \varepsilon_t, \tag{29}$$

where $T_0^\rho = T_0^\mu = 1$, $T_{m+1}^\rho = T_{m+1}^\mu = T$, the parameters during the first subperiods are $\rho^1 = \rho$ and $\mu^1 = \mu$, and the parameters during subperiods $j$ are $\rho^j = \rho^{j-1} + \Delta\rho_j$ for

$j = 2, 3, \ldots, m$ and $\mu^j = \mu^{j-1} + \Delta\mu_j$ for $j = 2, 3, \ldots, n$.